Aggregation Across Each Nation: Trade and Macroeconomic Dynamics*

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Abstract

We study the implications of trade aggregation for macroeconomic dynamics in workhorse open-economy models, deriving sufficient statistics for the impact of the functional form of the trade aggregator on the first-order dynamics of the model. With two countries, for given steady-state trade elasticities and expenditure shares, any aggregator that is homogeneous of degree one is equivalent to the widely used Armington aggregator at first order. Aggregators that are homogeneous of different degree are equivalent to a generalised Armington aggregator. With more countries, alternative aggregators affect macroeconomic dynamics through steady-state differences in bilateral trade elasticities across country-pairs, which Armington rules out.

Key Words: International Trade, Open-economy Macroeconomics, Armington Aggregator, Elasticity of Trade.
JEL Codes: F00, F10, F41.

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1 Introduction

Goods trade is central to New Open-Economy Macroeconomics (NOEM) models (Corsetti, 2008), playing an important role in the cross-border propagation of shocks. One of the most primitive assumptions of any international macroeconomics model is how domestic and foreign goods are bundled together to form aggregate goods. The structure of this aggregation has implications for how goods demand responds to changes in relative prices, and influences aggregate wealth and intertemporal consumption-savings decisions. Therefore, this aggregation is central to macroeconomic dynamics and our understanding of many features of the global economy. In this paper, we analytically derive sufficient statistics that summarise the impact of the trade aggregator on the first-order dynamics of workhorse NOEM models, and assess the implications of how trade aggregation is modelled for macroeconomic dynamics.

Our analysis is grounded in the class of workhorse multi-country NOEM models (e.g., Backus, Kehoe, and Kydland, 1992). Within this literature, it is widely understood that cross-border trade is crucial for the size and, in some cases, the sign of the international transmission of shocks (Corsetti, Dedola, and Leduc, 2008; Bodenstein, 2010, 2011), as well as the degree of business-cycle synchronisation (Johnson, 2014). However, while the importance of trade is well known, to date no studies have explored how the choice of trade aggregator impacts the dynamics of NOEM models.

We show that within the class of aggregators that are homogeneous of degree one, the implications of trade for the first-order dynamics of a NOEM model are entirely determined by two sets of sufficient statistics: the steady-state consumption expenditure shares and the steady-state elasticity of substitution. The first set captures the shares of expenditure on each good, which typically reflect the degree of 'home bias' in preferences—the idea that countries tend to spend proportionally more on domestic goods even if prices are symmetric. The second captures the elasticities of substitution between goods produced in different countries—the 'trade elasticity'—governing how relative demand responds to relative prices. Hence, our results show that for a given calibration of these two sets of objects, the precise functional form of the aggregator is irrelevant.

In a two-country setup, this result implies that any aggregator that is homogeneous of degree one is equivalent to a suitably parameterised Armington (1969) aggregator. The Armington aggregator, a Constant Elasticity of Substitution (CES) aggregator, is the 'go-to' aggregator in multi-country models due to its tractability and closed-form solutions. The aggregator function is summarised by two parameters, which in fact precisely correspond to the same two sets of sufficient statistics highlighted above: home bias and the trade elasticity. While the Armington aggregator is widely applied in NOEM models focused on the spillovers from macroeconomic shocks, a largely independent literature has put forward a set of alternative functional forms for trade aggregation.[1] Our sufficient-statistics result shows that any aggregator that is homogeneous of degree one is equivalent, at first order, to the Armington aggregator with its parameters calibrated to match steady-state expenditure shares and trade elasticities.

Extending our setup to include more than two countries, there is no longer just one single bilateral trade elasticity of substitution, but potentially different elasticities across each pair of country-goods. Our results show that the sufficient statistics for the first-order model dynamics now include the steady-state elasticity of substitution across every pair of country-goods. This is because the cross-border transmission of shocks in a multi-country model depends not only on the bilateral trade between two countries, but also their indirect linkages via third countries.

Since the Armington aggregator imposes that the elasticity is the same across all pairs of country-goods, an alternative aggregator can change the first-order dynamics of the model relative to Armington by allowing for different elasticities of substitution across different pairs of goods. This is important because the Armington aggregator is commonplace in multi-country quantitative models, calibrated with steady-state asymmetries across countries, which are often used for policy analysis. For instance, the International Monetary Fund's Global Integrated Monetary and Fiscal model (Laxton, Mursula, Kumhof, and Muir, 2010) features layered CES aggregation of domestic and foreign, consumption and investment, and final and intermediate goods. We show that the use of alternative aggregators in such cases widens the range of possible dynamics, by allowing bilateral trade elasticities to differ across different country pairs in steady state.

Finally, we extend the analysis to cases in which the aggregator is not homogeneous of degree one. The first-order dynamics of the model in this setting are summarised by a broader set of sufficient

[1]See, *inter alia*, Kimball (1995); Bergin and Feenstra (2000); Feenstra (2003); Arkolakis and Morlacco (2017); Feenstra (2018); Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2019); Jung, Simonovska, and Weinberger (2019); Fally (2022).

statistics: the steady-state consumption expenditure shares and steady-state bilateral elasticities as before, plus steady-state ratios related to the degree of homogeneity of the aggregator. Here, the first-order dynamics will change relative to the Armington aggregator, even in a two-country model, due to the difference in the degree of homogeneity. We propose a simple extension of the Armington aggregator, introducing one new parameter. This generalised Armington aggregator can parsimoniously replicate any aggregator that is homogeneous of arbitrary degree in a two-country setup. As before, in a setup with more than two countries, differences in steady-state bilateral elasticities of substitution can affect the dynamics of the model in a way that the generalised Armington aggregator cannot replicate.

Our paper contributes to a growing literature searching for sufficient statistics that govern equilibrium outcomes in multi-country and multi-sector models. Remarkably, the two sets of sufficient statistics that we identify are precisely the quantities Arkolakis, Costinot, and Rodriguez-Clare (2012) emphasise as sufficient for characterising the welfare implications of a class of trade models. This parallel arises despite important differences between our respective environments: our focus is on first-order macroeconomic dynamics in an intertemporal NOEM model, while Arkolakis et al. (2012) analyse welfare in a static setting à la Eaton and Kortum (2002).[2] Closest in spirit to our contribution, Baqaee and Farhi (2019) investigate the implications of 'Hulten's theorem' (Hulten, 1978) in a multi-sector open-economy setup. They assess the impact of sectoral shocks propagating through global production networks, and derive sufficient statistics in terms of the input-output structure of the economy. In contrast, our work focuses on the dynamics of the workhorse NOEM model, and the functional form of the trade aggregator.

Overall, our analytical results highlight the importance of two types of parameters for trade and macroeconomic dynamics. We show the specific settings in which the widely used Armington aggregator is sufficient to capture the impact of trade aggregation, and also highlight precisely how alternative aggregators can create scope for richer macroeconomic dynamics in more general settings.[3] Moreover, because the sufficient statistics we uncover have clear empirical counterparts, our theoretical findings justify continued focus on the estimation of trade elasticities and shares from micro data (e.g., Feenstra, Luck, Obstfeld, and Russ, 2018; Freeman,

[2] See also Arkolakis and Morlacco (2017), who analyse the properties of alternative, non-CES, demand functions.

[3] Our results are derived analytically from the linearised model, and so hold exactly at first order. Alternative aggregators may have additional effects at higher order, but by definition these effects will be small unless there is a non-linearity in the model, or if we consider large shocks or shocks to higher moments.

Larch, Theodorakopoulos, and Yotov, 2021).

The rest of the paper is organised as follows. Section 2 shows the main features of our generic model. Section 3 contains the general form of the core sufficient-statistics result. Section 4 explores its implications through a series of specific cases. Section 5 concludes.

2 Model Setup

We first present a generic multi-country NOEM model. For simplicity and analytical tractability we consider endowment economies, hence abstracting from production and assuming that only final consumption goods are traded across countries. We discuss the generality of our results at the end of the section.

In this workhorse model, the problem of the representative consumer can be split into *inter*temporal and *intra*temporal components. The intratemporal component is aggregator-specific, taking the aggregate choices from the intertemporal problem as given. The intertemporal household problem is analytically independent of the aggregation structure, and defines aggregate quantities in equilibrium. For this reason, our results are independent of the precise formulation of the intertemporal block of the model. This separability of intertemporal and intratemporal decisions, which is important for our main result, is a widely applied assumption in the class of NOEM models we study.

2.1 The Model

The model has N countries, indexed by $n = 1, 2, ..., N$. Time is discrete and infinite. In each time period t, each country n is endowed with a unique tradable good, denoted by $Y_t^{(n)}$, which is strictly positive. Variation in country-specific endowments is the sole source of uncertainty. Endowments are subject to stochastic mean-zero disturbances each period, which generate fluctuations around their mean value, denoted by $\overline{Y}^{(n)}$. Hence the steady state of the model is defined as the deterministic equilibrium with $Y_t^{(n)} = \overline{Y}^{(n)}$, $\forall\, t,\, n$.

Intertemporal Problem. The representative consumer in country n has expected lifetime utility:

$$U_t^{(n)} = \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau u\left(C_{t+\tau}^{(n)}\right) \right],$$

where $C_t^{(n)}$ denotes period-t aggregate consumption; $u : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable, strictly increasing and strictly concave function, with $\lim_{C\to 0} u'(C) = \infty$; and $\beta \in (0,1)$ is the discount factor.

Let $P_t^{(n)}$ denote the price of a unit of aggregate consumption in country n and $P_{i,t}^{(n)}$ the price of a unit of the country-i good in country n. The intertemporal budget constraint of the country-n representative consumer is:

$$\sum_{\tau=0}^{\infty} \left(P_{t+\tau}^{(n)} C_{t+\tau}^{(n)} - P_{n,t+\tau}^{(n)} Y_{t+\tau}^{(n)} \right) \le 0.$$

Without loss of generality, we consider complete international capital markets. We obtain the equilibrium risk-sharing conditions by equalising the optimality conditions for the representative households in two countries, n and n':

$$RER_{t+\tau}^{(n,n')} = \kappa^{(n,n')} u'\left(C_{t+\tau}^{(n')}\right) / u'\left(C_{t+\tau}^{(n)}\right), \quad \forall\, n' \neq n,$$

where $RER_t^{(n,n')} \equiv P_t^{(n')}/P_t^{(n)}$ denotes the real exchange rate of country n *vis-à-vis* country n', defined such that an increase in its value represents a depreciation for country n, and $\kappa^{(n,n')} > 0$ is a risk-sharing constant.

Assumption 1: In steady state, there is bilateral balanced trade between every pair of countries.

The risk-sharing constant, $\kappa^{(n,n')}$, can be set so as to ensure that the risk-sharing condition is satisfied at the steady state under Assumption 1. Specifically, $\kappa^{(n,n')} = 1$ when countries are symmetric at steady state, but it will differ from unity when there are steady-state asymmetries across countries. As long as Assumption 1 is satisfied, our results hold under financial autarky or other forms of incomplete markets.

In equilibrium, the intertemporal optimisation of representative consumers in each country, and the risk-sharing conditions, pins down the sequence of $C_t^{(n)}$ and $RER_t^{(n,n')}$ given the sequence of endowments.

Intratemporal Problem. Aggregate consumption of households in country n is formed of goods produced in all N countries, using the aggregator function $f : \mathbb{R}_+^N \to \mathbb{R}$, such that:

$$C_t^{(n)} \equiv f\left(\mathbf{c}_t^{(n)}\right) \tag{1}$$

where $\mathbf{c}_t^{(n)} = \left[c_{1,t}^{(n)}, c_{2,t}^{(n)}, ..., c_{N,t}^{(n)}\right]'$ denotes the $N \times 1$ vector of consumption levels, with $c_{i,t}^{(n)}$ denoting the representative country-n household's consumption of goods from country i.

The time-t intratemporal problem of the representative household minimises total expenditure, taking as given the level of aggregate consumption, $C_t^{(n)}$, from the intertemporal optimisation and the prices, $P_{i,t}^{(n)}$, as defined above.

Assumption 2: The law of one price (LOOP) holds, such that $P_{i,t}^{(n)} = P_{i,t}$, $\forall\, n$.

Using Assumption 2, the intratemporal problem of the household in country n can be written as:

$$\min_{\mathbf{c}_t^{(n)}} \sum_{i=1}^{N} P_{i,t} c_{i,t}^{(n)} \quad \text{subject to} \quad C_t^{(n)} = f\left(\mathbf{c}_t^{(n)}\right).$$

Assumption 3: The function f is continuous, twice differentiable and strictly quasi-concave.

Under Assumption 3, the solution to the intratemporal problem exists, is unique, and is defined by $(N-1)$ relative-demand functions:

$$\frac{f_{i,t}^{(n)}}{f_{N,t}^{(n)}} = \frac{p_{i,t}^{(n)}}{p_{N,t}^{(n)}} \quad \text{for } i = 1, 2, ..., N-1. \tag{2}$$

where $f_{i,t}^{(n)} \equiv \partial f(\mathbf{c}_t^{(n)})/\partial c_{i,t}^{(n)}$,[4] and $p_{i,t}^{(n)} \equiv P_{i,t}/P_t^{(n)}$ is the price of each good relative to the aggregate consumer price index, $P_t^{(n)}$. This index can be defined as the price per unit of aggregate consumption using:

$$P_t^{(n)} C_t^{(n)} = \sum_{i=1}^{N} P_{i,t} c_{i,t}^{(n)}$$

[4] We will denote second derivatives by: $f_{ik,t}^{(n)} = \partial f_{i,t}^{(n)}/\partial c_{k,t}^{(n)}$, $\forall\, i, k$.

so that:

$$C_t^{(n)} = \sum_{i=1}^{N} p_{i,t}^{(n)} c_{i,t}^{(n)} \tag{3}$$

World equilibrium in goods markets is given by:

$$Y_t^{(n)} \geq \sum_{i=1}^{N} c_{n,t}^{(i)} \quad \text{for all } n. \tag{4}$$

In equilibrium, the intratemporal optimisation in each country, and the goods market clearing conditions, define the trade quantities $\mathbf{c}_t^{(n)}$ and relative prices $\mathbf{p}_t^{(n)}$, given the aggregate variables from the intertemporal problem.

2.2 Generality

In the above exposition, we have highlighted Assumptions 1-3, which are necessary for our key results. While we have specified all of the other aspects of our model, including making additional assumptions, our main theorem holds in more general settings so long as those three assumptions are satisfied.

As discussed at the start of this section, the separation of the intratemporal and intertemporal blocks of the model means that the results about the trade aggregator are invariant to the specification of the intertemporal block. Most notably, the results would continue to hold if we removed the simplifying assumption of exogenous endowments, and instead introduced a production sector. So long as the intratemporal demand for each type of good remains separate to the production side of the model, our results would still hold. Similarly, since intertemporal decisions about aggregate consumption and saving can be separated from the intratemporal aggregation problem, the exact structure and completeness of international financial markets has no implications for our results, so long as the steady state satisfies Assumption 1.

Moreover, we have focused here on the *consumption* aggregator, and so we have highlighted the separation of the intratemporal component of the consumer problem. The same results would hold in more complex models with specific intertemporal variation in the trade elasticity (Cooley and Quadrini, 2003) or aggregators for other types of goods, such as intermediate inputs or investment goods, so long as the optimal composition of these goods between domestic and foreign goods remains an intratemporal choice.

3 Sufficient Statistics for the Aggregator

The key question of this paper is how the specific choice of functional form for f affects the model's equilibrium macroeconomic dynamics. Our main result is summarised by Theorem 1:

Theorem 1 *Under Assumptions 1-3, the effect of the aggregator function f on the first-order dynamics of the model is captured entirely by the following sufficient statistics, where overlines represent the steady-state values of variables and functions thereof:*

(i) the share of consumption expenditure on each good:

$$\overline{\alpha}_i^{(n)} \equiv \frac{\overline{p}_i^{(n)}\overline{c}_i^{(n)}}{\overline{C}^{(n)}} \quad \textit{for } i = 1, 2, ..., N$$

(ii) the elasticities of substitution between each pair of goods:

$$\overline{\Phi}_{i,j}^{(n)} \equiv \frac{\partial \ln \left(\overline{c}_i^{(n)}/\overline{c}_j^{(n)}\right)}{\partial \ln \left(\overline{f}_j^{(n)}/\overline{f}_i^{(n)}\right)} \quad \textit{for } i,\ j = 1, 2, ..., N,\ i \neq j$$

(iii) the ratio $\overline{\mathcal{H}}^{(n)}$, defined as:

$$\overline{\mathcal{H}}^{(n)} \equiv \mathcal{H}(\overline{\mathbf{c}}^{(n)}) = \frac{\sum_{i=1}^{N} \overline{f}_i^{(n)}\overline{c}_i^{(n)}}{f(\overline{\mathbf{c}}^{(n)})}$$

(iv) the ratios $\overline{\mathcal{H}}_i^{(n)}$ for each good, defined as:

$$\overline{\mathcal{H}}_i^{(n)} \equiv \mathcal{H}_i(\overline{\mathbf{c}}^{(n)}) = \frac{\sum_{k=1}^{N} \overline{f}_{ik}^{(n)}\overline{c}_k^{(n)}}{f_i(\overline{\mathbf{c}}^{(n)})} \quad \textit{for } i = 1, 2, ..., N$$

for each country $n = 1, 2, ..., N$.

Proof: First, notice that equations (1)-(4) are the only model equations affected by f, and the consumption levels, $\mathbf{c}_t^{(n)}$, that it defines. The remaining equations are independent of the aggregator by definition. We prove the theorem by showing that the first-order approximation of these four equations only depends on f through the steady-state quantities described above. Full derivations are in Appendix A. $\square$

The elasticities of substitution, consumption expenditure shares, and the ratios $\mathcal{H}(.)$ and $\mathcal{H}_i(.)$ are generically functions of the variables of the model and therefore can vary dynamically. However, Theorem 1 states that the first-order dynamics of the model depend only on the *steady-state* values of these objects.

Before unpacking the theorem's implications, it is useful to say a few words on $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$, as they may be unfamiliar. We can better understand these statistics by considering a specific class of aggregators: homogeneous functions. We refer to functions that are homogeneous of degree h as being HOD(h). The result for this group is summarised by Corollary 1:

Corollary 1 *If the aggregator function, f, is homogeneous of degree h, the first-order dynamics of the model are captured by the following sufficient statistics: $\overline{\alpha}_i^{(n)}$, $\overline{\Phi}_{i,j}^{(n)}$ $\forall i, j, n, i \neq j$, and h.*

Proof: If f is HOD(h), then its partial derivatives are HOD($h-1$). By Euler's theorem, $\overline{\mathcal{H}}^{(n)} = h$ and $\overline{\mathcal{H}}_i^{(n)} = (h-1)$, $\forall\, i$. Hence, h becomes the sufficient statistic for $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$. Full derivations are in Appendix B. $\square$

When comparing across aggregators that are homogeneous of the same degree, Corollary 1 has implications that are summarised in Corollary 2:

Corollary 2 *All aggregators that are homogeneous of the same degree will imply the same first-order dynamics, for given $\overline{\alpha}_i^{(n)}$ and $\overline{\Phi}_{i,j}^{(n)}$ $\forall i, j, n, i \neq j$.*

Proof: This follows directly from Corollary 1. $\square$

4 Implications of the Theorem

In this section, we explore the implications of Theorem 1 and Corollaries 1 and 2, by considering a series of separate cases.

4.1 Homothetic Preferences

A basic assumption in most economic models is that preferences are homothetic. A homothetic function is a monotonic transformation of a function that is HOD(1). Therefore, if the utility function, $u(C_t^{(n)})$, is a monotonic increasing function of aggregate consumption, then utility is homothetic with respect to $\mathbf{c}_t^{(n)}$ if the aggregator function f is HOD(1).

Corollary 2 implies that, within the class of HOD(1) aggregators, the sufficient statistics for the first-order model dynamics are just the steady-state elasticities of substitution and expenditure shares. Since this class includes the Armington aggregator, this means that any alternative HOD(1) aggregator, with the same steady-state elasticities of substitution and expenditure shares, will result in the same macroeconomic dynamics as the Armington aggregator.

We unpack these implications by comparing the Armington aggregator to the Kimball (1995) aggregator, an alternative and widely used HOD(1) aggregator. We do so in two steps: first considering the two-country case, then extending our analysis to more than two countries.

4.1.1 Two Countries

If $N = 2$, all aggregators that are HOD(1) are equivalent at first order to the Armington aggregator with the same steady-state elasticity and home bias.

Armington Aggregator. The Armington aggregator for a two-country model is given by:

$$C_t^{(n)} \equiv f(c_{1,t}^{(n)}, c_{2,t}^{(n)}) = \left(a_1^{(n)\frac{1}{\phi}} c_{1,t}^{(n)\frac{\phi-1}{\phi}} + \left(1 - a_1^{(n)}\right)^{\frac{1}{\phi}} c_{2,t}^{(n)\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}} \qquad \text{for } n = 1, 2$$

and yields the familiar relative demand functions:

$$\frac{c_{1,t}^{(n)}}{c_{2,t}^{(n)}} = \frac{a_1^{(n)}}{1 - a_1^{(n)}} \left(\frac{p_{2,t}^{(n)}}{p_{1,t}^{(n)}} \right)^{\phi} \qquad \text{for } n = 1, 2$$

where ϕ is the constant elasticity of substitution between the two goods, and $a_1^{(n)}$ captures the degree of home bias in each country, which maps into the steady-state consumption expenditure shares, $\overline{\alpha}_1^{(n)}$ and $\overline{\alpha}_2^{(n)} = (1 - \overline{\alpha}_1^{(n)})$.[5]

[5] In a symmetric steady state, in which the prices of the two goods are equal, then $\overline{\alpha}_1^{(n)} = a_1^{(n)}$, but outside of symmetry this mapping will depend on the steady-state relative prices, with $\overline{\alpha}_1^{(n)} = a_1^{(n)}(\overline{P}_1^{(n)}/\overline{P}^{(n)})^{1-\phi}$.

The first-order approximation of these equations is given by:

$$\widetilde{C}_t^{(n)} = \overline{\alpha}_1^{(n)}\widetilde{c}_{1,t}^{(n)} + (1-\overline{\alpha}_1^{(n)})\widetilde{c}_{2,t}^{(n)} \tag{5}$$

$$\widetilde{c}_{1,t}^{(n)} - \widetilde{c}_{2,t}^{(n)} = \phi\left(\widetilde{p}_{2,t}^{(n)} - \widetilde{p}_{1,t}^{(n)}\right) \tag{6}$$

for $n = 1, 2$, where $\widetilde{x}_t$ is the percentage deviation of variable x from its steady state $\bar{x}$.

These equations illustrate how the two parameters of the Armington aggregator enter the linearised model equations.

Kimball Aggregator. Consider Kimball (1995)'s aggregator, where aggregate consumption $C_t^{(n)}$ is implicitly defined by:

$$1 = b_1^{(n)}\Upsilon\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)}C_t^{(n)}}\right) + b_2^{(n)}\Upsilon\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)}C_t^{(n)}}\right) \quad \text{for } n = 1, 2 \tag{7}$$

where $b_2^{(n)} \equiv (1 - b_1^{(n)})$, and $\Upsilon(.)$ is a function that satisfies $\Upsilon(1) = 1$, $\Upsilon'(.) > 0$ and $\Upsilon''(.) > 0$. It can be seen from this implicit definition of $C_t^{(n)}$ that aggregate consumption is HOD(1) with respect to consumption of country-specific goods.

We follow Klenow and Willis (2016) and specify the function $\Upsilon(.)$ as: [6]

$$\Upsilon(x) = 1 + (\sigma - 1)\exp(\epsilon^{-1})\epsilon^{\frac{\sigma}{\epsilon}-1}\left(\Gamma\left(\frac{\sigma}{\epsilon}, \frac{1}{\epsilon}\right) - \Gamma\left(\frac{\sigma}{\epsilon}, \frac{x^{\frac{\epsilon}{\sigma}}}{\epsilon}\right)\right)$$

where:

$$\Gamma(u, z) = \int_z^{+\infty} s^{u-1}\exp(-s)\,\mathrm{d}s.$$

This aggregator is defined by three parameters: σ, ϵ and $b_1^{(n)}$. The latter is a familiar home-bias parameter, which maps into consumption shares, while σ and ϵ pin down the elasticity of substitution.

[6] Alternative formulations of the Kimball (1995) aggregator specify $\Upsilon(.)$ differently, e.g. Lindé and Trabandt (2018) use a Dotsey and King (2005). The specific choice of functional form is irrelevant for our results.

To see this, consider the relative demand functions from the household's intratemporal problem:

$$\frac{p_{1,t}^{(n)}}{p_{2,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)} C_t^{(n)}}\right)}{\Upsilon'\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)} C_t^{(n)}}\right)} \quad \text{for } n = 1, 2 \tag{8}$$

From this, we can define the consumption shares and elasticity of substitution:

$$\alpha_{1,t}^{(n)} = \frac{p_{1,t}^{(n)} c_{1,t}^{(n)}}{p_{1,t}^{(n)} c_{1,t}^{(n)} + p_{2,t}^{(n)} c_{2,t}^{(n)}} = \frac{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)} C_t^{(n)}}\right) c_{1,t}^{(n)}}{\Upsilon'\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)} C_t^{(n)}}\right) c_{1,t}^{(n)} + \Upsilon'\left(\frac{c_{2,t}^{(n)}}{b_2^{(n)} C_t^{(n)}}\right) c_{2,t}^{(n)}} \tag{9}$$

$$\Phi_{1,2,t}^{(n)} = \sigma \left(1 + \frac{\alpha_{1,t}^{(n)}}{\alpha_{2,t}^{(n)}}\right) \left[\left(\frac{c_{1,t}^{(n)}}{b_1^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_{1,t}^{(n)}}{\alpha_{2,t}^{(n)}} \left(\frac{c_{2,t}^{(n)}}{b_2^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right]^{-1} \tag{10}$$

for $n = 1, 2$.[7]

Equation (10) illustrates the key property of Kimball preferences: the elasticity of substitution is no longer constant. Notice that as $\epsilon \to 0$, $\Phi_{1,2,t}^{(n)} \to \sigma$, implying that Kimball nests CES, with elasticity σ, as a limit case.

More generally, when $\epsilon > 0$, the elasticity of substitution depends on the relative consumption levels. When consumption of good 1 is low relative to good 2, then the elasticity of substitution rises, and a decrease in the relative price of good 1 leads to a larger substitution towards good 1. Conversely, when the consumption of good 1 is high relative to good 2, then the elasticity of substitution falls, and a decrease in the relative price of good 1 leads to a smaller substitution towards good 1. The parameter ϵ controls the curvature of the demand function: the higher its value, the more the elasticity varies with the relative consumption levels.[8]

Comparing Armington and Kimball. Despite these additional mechanisms in the Kimball aggregator, the application of Corollary 2 to this case tells us that a Kimball trade aggregator implies the same first-order dynamics as the Armington aggregator. To see why, we take the first-order approximation of the implicit definition of aggregate consumption, equation (7), and

[7] Full derivations are provided in Appendix C.

[8] See Appendix D for more details on the properties of the Kimball aggregator.

the relative demand function, equation (8):

$$\widetilde{C}_t^{(n)} = \overline{\alpha}_1^{(n)} \widetilde{c}_{1,t}^{(n)} + (1 - \overline{\alpha}_1^{(n)}) \widetilde{c}_{2,t}^{(n)} \tag{11}$$

$$\widetilde{c}_{1,t}^{(n)} - \widetilde{c}_{2,t}^{(n)} = \overline{\Phi}_{1,2}^{(n)} \left(\widetilde{p}_{2,t}^{(n)} - \widetilde{p}_{1,t}^{(n)} \right) \tag{12}$$

for $n = 1, 2$, where $\overline{\alpha}_1^{(n)}$ and $\overline{\Phi}_{1,2}^{(n)}$ are the steady-state values of the consumption share and elasticity of substitution defined in equations (9) and (10).

This shows us that the linearised equations only depend on the *steady-state* consumption shares and the *steady-state* elasticity of substitution. The parameters of the Kimball aggregator, including the curvature parameter ϵ, only matter insofar as they pin down these two steady-state values. Importantly, then, despite the fact that $\epsilon > 0$ allows for the elasticity of substitution to vary dynamically, these dynamics do not enter the linearised model equations.

Thus, for a given parameterisation of the Kimball aggregator, we can set the Armington parameters, $a_1^{(n)}$ to match the same $\overline{\alpha}_1^{(n)}$, and $\phi = \overline{\Phi}_{1,2}^{(n)}$, and we see that equations (11) and (12) are exactly equivalent to the linearised equations under CES, (5) and (6). In other words, the first-order dynamics of the two-country model with the Kimball aggregator are equal to those of a CES specification, for given steady-state consumption shares and steady-state elasticity of substitution.

4.1.2 $N > 2$ Countries

If $N > 2$, the specific form of the aggregator is relevant only to the extent that the bilateral elasticities of substitution across different pairs of goods are different in steady state.

Armington Aggregator. In the N-good setting, the Armington aggregator in country n is:

$$C_t^{(n)} = f(c_{1,t}^{(n)}, \ldots c_{N,t}^{(n)}) = \left(\sum_{i=1}^{N} a_i^{(n)\frac{1}{\phi}} c_{i,t}^{(n)\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}} \qquad \text{for all } n$$

where $\sum_{i=1}^{N} a_i^{(n)} = 1$.

The relative-demand functions are given by:

$$\frac{c_{i,t}^{(n)}}{c_{N,t}^{(n)}} = \frac{a_i^{(n)}}{a_N^{(n)}} \left(\frac{p_{N,t}^{(n)}}{p_{i,t}^{(n)}} \right)^{\phi} \quad \text{for all } i \neq N$$

for all n. This leads to the following linearised equations $\forall\ n$:

$$\widetilde{C}_t^{(n)} = \sum_{i=1}^{N} \overline{\alpha}_i^{(n)} \widetilde{c}_{i,t}^{(n)} \tag{13}$$

$$\widetilde{c}_{i,t}^{(n)} - \widetilde{c}_{N,t}^{(n)} = \phi \left(\widetilde{p}_{N,t}^{(n)} - \widetilde{p}_{i,t}^{(n)} \right) \quad \text{for all } i \neq N \tag{14}$$

The pairwise elasticities of substitution between any two goods are given by the same parameter ϕ. To see how this property affects the comparison with more general aggregators, we reconsider the Kimball aggregator.

Kimball Aggregator. The implicit definition of the N-good Kimball aggregator is:

$$1 = \sum_{i=1}^{N} b_i^{(n)} \Upsilon \left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}} \right) \qquad \text{for all } n$$

where $\sum_{i=1}^{N} b_i^{(n)} = 1$ and the function $\Upsilon(.)$ is defined as in Section 4.1.1.

The resulting relative-demand functions are then:

$$\frac{p_{i,t}^{(n)}}{p_{N,t}^{(n)}} = \frac{\Upsilon' \left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}} \right)}{\Upsilon' \left(\frac{c_{N,t}^{(n)}}{b_N^{(n)} C_t^{(n)}} \right)} \quad \text{for all } i \neq N$$

$\forall\ n$.

As in the two-country case, we compute the consumption shares and the bilateral elasticities of substitution:

$$\alpha_{i,t}^{(n)} = \frac{p_{i,t}^{(n)} c_{i,t}^{(n)}}{\sum_{j=1}^{N} p_{j,t}^{(n)} c_{j,t}^{(n)}} = \frac{c_{i,t}^{(n)} \Upsilon' \left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}} \right)}{\sum_{j=1}^{N} c_{j,t}^{(n)} \Upsilon' \left(\frac{c_{j,t}^{(n)}}{b_j^{(n)} C_t^{(n)}} \right)} \quad \text{for all } i$$

$$\Phi_{i,j,t}^{(n)} = \sigma \left(1 + \frac{\alpha_{i,t}^{(n)}}{\alpha_{j,t}^{(n)}} \right) \left[\left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}} \right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_{i,t}^{(n)}}{\alpha_{j,t}^{(n)}} \left(\frac{c_{j,t}^{(n)}}{b_j^{(n)} C_t^{(n)}} \right)^{\frac{\epsilon}{\sigma}} \right]^{-1} \quad \text{for all } i, j, \; i \neq j \tag{15}$$

$\forall \; n$.[9]

As before, the elasticity of substitution depends on the relative consumption levels. As well as allowing the elasticity to vary over time, we see that the elasticity can differ for different pairs of goods, depending on asymmetries between countries. By examining equation (15), we can see that the elasticities between two pairs of goods, $\{i, j\}$ and $\{i, l\}$, will be equal if and only if one of three conditions holds: (i) $\epsilon = 0$, in which case we are back to the CES aggregator with $\Phi_{i,j,t}^{(n)} = \sigma$, $\forall \; i, j, t$; (ii) $\alpha_{i,t}^{(n)} = 0$, implying that good i is not consumed at all; or, most importantly, (iii) $b_j^{(n)} = b_l^{(n)}$ and $c_{j,t}^{(n)} = c_{l,t}^{(n)}$, such that consumption shares are equal across goods. Ignoring the trivial cases (i) and (ii), we therefore see that the Kimball aggregator implies that the elasticities across different pairs of goods will be different, unless there is perfect symmetry across all countries.

Comparing Armington and Kimball. To see how these differences in elasticities across country-pairs affect the dynamics of the model, we again take the first-order approximation of

[9] Full derivations are in Appendix C.

the aggregator and relative-demand functions in the N-country setting:

$$\widetilde{C}_t^{(n)} = \sum_{i=1}^{N} \overline{\alpha}_i^{(n)} \widetilde{c}_{i,t}^{(n)} \tag{16}$$

$$\begin{aligned}\widetilde{p}_{i,t}^{(n)} - \widetilde{p}_{N,t}^{(n)} =& \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k^{(n)} \sum_{l=1, l\neq k}^{N} \left[\overline{\alpha}_k^{(n)} \left(\left(\overline{\Phi}_{Nl}^{(n)}\right)^{-1} - \left(\overline{\Phi}_{il}^{(n)}\right)^{-1} \right) + \overline{\alpha}_l^{(n)} \left(\left(\overline{\Phi}_{ik}^{(n)}\right)^{-1} - \left(\overline{\Phi}_{Nk}^{(n)}\right)^{-1} \right) \right. \\ &\left. + \frac{\overline{\alpha}_k^{(n)} \overline{\alpha}_l^{(n)}}{\overline{\alpha}_N^{(n)}} \left(\left(\overline{\Phi}_{Nl}^{(n)}\right)^{-1} - \left(\overline{\Phi}_{Nk}^{(n)}\right)^{-1} \right) + \frac{\overline{\alpha}_k^{(n)} \overline{\alpha}_l^{(n)}}{\overline{\alpha}_i^{(n)}} \left(\left(\overline{\Phi}_{ik}^{(n)}\right)^{-1} - \left(\overline{\Phi}_{il}^{(n)}\right)^{-1} \right) \right] \\ & \text{for all } i \neq N \end{aligned} \tag{17}$$

$\forall\ n$.[10] We see that the additional countries create additional terms in the linearised relative-demand functions, capturing the indirect substitution between goods i and N via goods k, l.

Importantly, when we have perfect symmetry across countries in steady state, such that $\overline{\Phi}_{i,j}^{(n)} = \overline{\Phi}_{i,l}^{(n)}$, $\forall\ i, j, l$, then these additional terms disappear from all relative-demand functions. This symmetry across country pairs is why these terms were absent for the Armington aggregator in equations (13) and (14). In this symmetric case, therefore, we can again replicate the first-order dynamics from the Kimball aggregator using an Armington aggregator by matching the steady-state consumption shares, and setting ϕ to match this common elasticity of substitution.

However, if we allow for steady-state asymmetries across countries, these additional terms in (16) and (17) will create first-order effects that cannot be captured by an Armington aggregator. Notice that it is again only the steady-state values of the elasticities that enter the linearised equations, and not any dynamic variation in the elasticity. Nonetheless, the Kimball aggregator allows us to map steady-state asymmetries in, say, endowments, into differences in elasticities of substitution, which then impact the dynamics of the model.

Numerical Exercise with $N = 3$. To illustrate these effects of using the Kimball aggregator, we consider a three-country version of our model. We label the countries $n = \{H, F, R\}$ and consider our results from the perspective of the Home country, H. For this stylised exercise, we set the discount factor $\beta = 0.99$, and assume the instantaneous utility function $u(.)$ has the

[10]We use the convention that $\left(\Phi_{i,i,t}^{(n)}\right)^{-1} = 0$, for all i, n and t.

constant relative risk aversion form, with a coefficient of relative risk aversion of 2.

We set the parameters of the Armington and Kimball aggregators so that they are equivalent in the symmetric steady state: $a_i^{(i)} = b_i^{(i)} = 0.7$ and $a_j^{(i)} = b_j^{(i)} = 0.15$, $\forall\, i, j \in \{H,\, F,\, R\}$, $j \neq i$. This implies that in the symmetric steady state, the domestic expenditure share is 70%, and the remaining expenditure share is split equally across the two foreign countries. We set $\phi = \sigma = 1.5$, so that, under symmetry, $\overline{\Phi}_{i,j}^{(n)} = 1.5$, $\forall\, i, j, n \in \{H,\, F,\, R\}$, $j \neq i$, independently of ϵ. We then keep these parameters fixed in the asymmetric setting, allowing the elasticities and expenditure shares to vary, and compare them across different values of ϵ.

To illustrate the role of Kimball aggregation, we depart from symmetry by assuming that the steady-state endowment of country H is smaller than the endowment of F and R. Normalising these values, we set $\overline{Y}^{(F)} = \overline{Y}^{(R)} = 1$ and $\overline{Y}^{(H)} = 0.5$. Table 1 shows the implied values of the steady-state objects of interest, across different values of ϵ, where the $\epsilon = 0$ column corresponds to the Armington case.

Table 1: Steady-State Expenditure Shares and Elasticities of Substitution Under Kimball

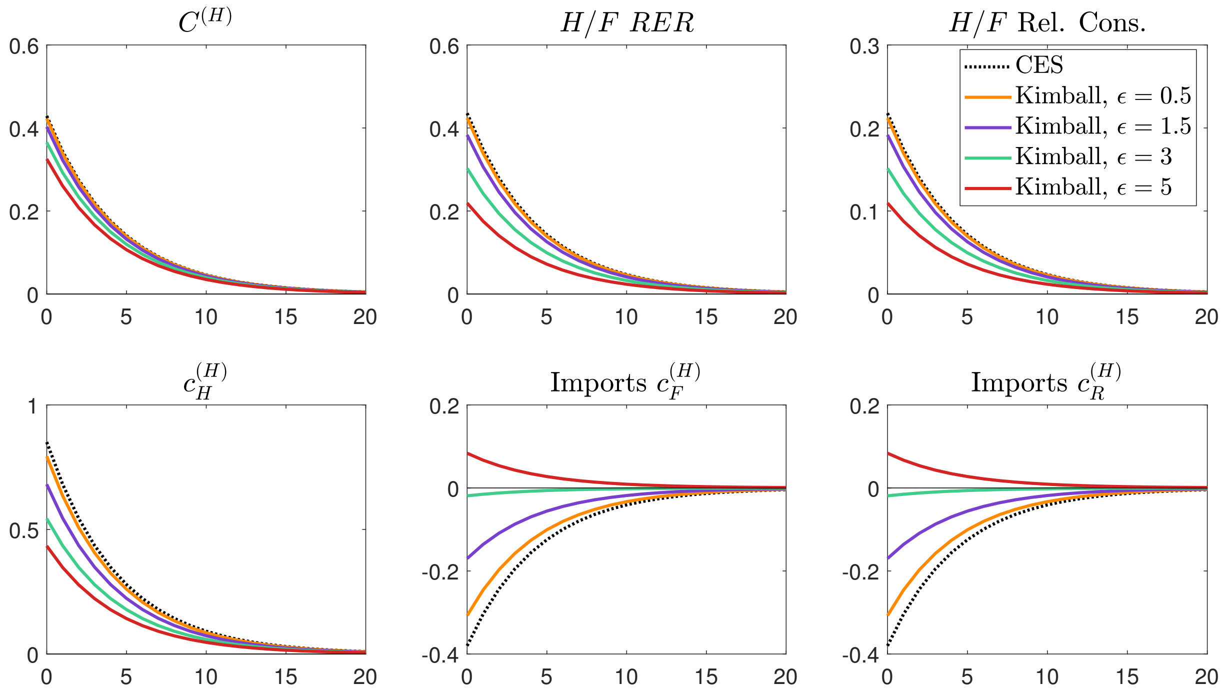
	Symmetry	Asymmetry				
		$\epsilon = 0$	$\epsilon = 0.5$	$\epsilon = 1.5$	$\epsilon = 3$	$\epsilon = 5$
$\bar{p}_H^{(H)} / \bar{p}_F^{(H)}$	1	1.48	1.46	1.41	1.35	1.28
$\bar{c}_H^{(H)} / \bar{c}_F^{(H)}$	1	3.84	3.94	4.12	4.31	4.45
$\bar{\alpha}_H^{(H)}$	70	65.74	66.32	67.31	68.29	68.99
$\bar{\alpha}_F^{(H)}$	15	17.13	16.84	16.35	15.86	15.51
$\bar{\Phi}_{H,F}^{(H)}$	1.5	1.50	1.37	1.17	0.99	0.86
$\bar{\Phi}_{F,R}^{(H)}$	1.5	1.50	1.33	1.09	0.89	0.76

Note: Due to the symmetry between F and R, $\bar{p}_F^{(H)} = \bar{p}_R^{(H)}$, $\bar{\alpha}_F^{(H)} \equiv \bar{\alpha}_R^{(H)}$ and $\bar{\Phi}_{H,F}^{(H)} \equiv \bar{\Phi}_{H,R}^{(H)}$.

Across all values of ϵ, the reduction in the supply of country-H goods increases its steady-state relative price above 1. Given that the Home good's relative international price is now higher, Home agents consume a higher share of their domestic good. Reflecting this high relative consumption, and in line with the mechanism explained in Section 4.1.1, the Home consumer's elasticity of substitution is smaller than in the Armington case, and declining in ϵ.

Figure 1 shows impulse response functions to a 1% increase in Home endowment starting from the asymmetric steady state, for different values of ϵ. After a positive endowment shock, the aggregate consumption in the Home country, $C^{(H)}$, always increases, more so with CES or

Figure 1: IRFs to a 1% endowment shock in the Home country, asymmetric 3-country case



Note: Home (H) country is assumed to be a small economy with steady-state endowment of 0.5, while the two other countries, (F) and (R), are large (endowment of 1). Dotted lines represent the CES responses with $\phi = 1.5$, while each solid line represents responses for different values of the Kimball 'curvature' parameter ϵ, with $\sigma = 1.5$. All consumers have symmetric preferences with home bias.

more convex Kimball preferences ($\epsilon < \sigma$), while the Home real exchange rate depreciates as the relative price of the Home good decreases.

More interestingly, the Home consumer's consumption responses for goods F and R change qualitatively with the curvature of Kimball preferences. With CES, and Kimball with low ϵ, the rise in Home consumption comes with a decline in their imports, as they switch towards the now-cheaper Home good. In other words, the response of trade quantities comoves negatively with the aggregate and domestic consumption. However, with large ϵ, this is no longer the case. The intuition behind this is as follows. When ϵ is larger, given that the Home consumers already consume a large quantity of the Home good, the steady-state elasticity of substitution between H and F goods, $\overline{\Phi}^{(H)}_{H,F}$, is relatively low. So the Home consumer switches towards the cheaper Home good more slowly. They will rather use their additional endowment to consume more of the F and R goods, financing it by selling Home goods. This triggers an increase in the imports of F and R goods when ϵ is high enough—$\epsilon = 5$ in Figure 1—such that trade comoves positively with aggregate consumption.

This example has illustrated how the use of Kimball in an $N > 2$ setting, with asymmetries across countries in the steady state, can both quantitatively and qualitatively impact macroeconomy dynamics and the transmission of shocks.[11]

4.2 Non-Homothetic Preferences

Theorem 1 and its corollaries also have implications for non-homothetic, or non-HOD(1), preferences.

Notice that non-HOD(1) can either mean HOD(h), for $h \neq 1$, or non-homogeneous. Theorem 1, which presented our result in the most general form, can be applied directly to the latter cases, meaning that all steady-state ratios $\overline{\mathcal{H}}^{(n)}$ and $\overline{\mathcal{H}}_i^{(n)}$, for all $i = 1, 2, ...N$, will affect the model's dynamics at first order. For a given parameterisation of a given non-homogeneous aggregator, their values can be derived. Theorem 1 then tells us that any differences in the macroeconomic dynamics implied by non-homogeneous aggregators will be explained by the values of these sufficient statistics.

In the rest of this section, we focus on the cases where the aggregator is HOD(h), for $h \neq 1$. Here, Corollary 1, by focusing specifically on homogeneous functions, shows that the dynamics only depend on h. It is useful to again consider the two cases.

4.2.1 Two Countries

If $N = 2$, any HOD(h) aggregator, $h \in \mathbb{R}$, is equivalent at first order to a generalised Armington-style aggregator that is HOD(h), with the same steady-state elasticity and home bias.

We can define a HOD(h) generalisation of the Armington aggregator in the 2-good setting as:

$$C_t^{(n)} \equiv f(c_{1,t}^{(n)}, c_{2,t}^{(n)}) = \left(a_1^{(n)\,\frac{1}{\phi}} c_{1,t}^{(n)\,\frac{\phi-1}{\phi}} + \left(1 - a_1^{(n)}\right)^{\frac{1}{\phi}} c_{2,t}^{(n)\,\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}h}$$

It is straightforward to show that this function is homogeneous of degree h. Then, by Euler's theorem, using the definitions from Theorem 1, $\mathcal{H}(c_{1,t}^{(n)}, c_{2,t}^{(n)}) = h$, and $\mathcal{H}_1(c_{1,t}^{(n)}, c_{2,t}^{(n)}) =$

[11] In Appendix E, we also consider a nested-CES structure—i.e. a layered sequence of two-good CES aggregators that recursively adds each country-good to the current bundle. We show that this specification, while retaining some of the tractability of the Armington aggregator, is not sufficiently flexible to generically replicate alternative aggregators, because it has the degrees of freedom to match all but one bilateral trade elasticity, leaving one sufficient statistic unmatched.

$\mathcal{H}_2(c_{1,t}^{(n)}, c_{2,t}^{(n)}) = (h-1)$, $\forall\ t, n$.

Note that the terms relating to h cancel out in the relative-demand function, so that the generalised aggregator leads to the same demand functions as in the standard case. Therefore, the parameters ϕ and $a_1^{(n)}$ have the same interpretation as before, and h is a free parameter that determines the degree of homogeneity.

This means that any model that uses an alternative HOD(h) aggregator can be mapped parsimoniously into this generalised Armington aggregator by setting the parameters ϕ and $a_1^{(n)}$ to match the steady-state elasticity of substitution and consumption shares, as before, and setting h equal to the same degree of homogeneity. Notice that, while these results show that deviating from HOD(1) aggregators can affect the first-order dynamics, even with $N = 2$, they also specify that the first-order effect of any HOD(h) aggregator relative to the standard Armington model is determined entirely by a single parameter, h.

4.2.2 $N > 2$ Countries

If $N > 2$, alternative HOD(h) aggregators can create differences with respect to the N-good generalised HOD(h) Armington aggregator, by allowing bilateral elasticities of substitution to be different across different pairs of goods in steady state.

In the N-good setting, the generalised HOD(h) Armington aggregator can be defined as:

$$C_t^{(n)} = f(c_{1,t}^{(n)}, \dots c_{N,t}^{(n)}) = \left(\sum_{i=1}^{N} a_i^{(n)\,\frac{1}{\phi}} c_{i,t}^{(n)\,\frac{\phi-1}{\phi}} \right)^{\frac{\phi}{\phi-1}h}$$

where $\sum_{i=1}^{N} a_i^{(n)} = 1$. Following the same logic as above, $\mathcal{H}(c_{1,t}^{(n)}, \dots c_{N,t}^{(n)}) = h$, and $\mathcal{H}_i(c_{1,t}^{(n)}, \dots c_{N,t}^{(n)}) = (h-1)$ for all i, t and n.

The same reasoning as the HOD(1) case can be applied here, again with the addition that the parameter h is chosen correctly. Note, once again, that the additional mechanism that alternative HOD(h) aggregators bring when $N > 2$ is only through the cross-elasticity differences in steady state.[12]

[12] These differences are also the ones that prevent matching a HOD(h) aggregator with a generalised nested-CES specification, using the generalised Armington.

5 Conclusions

We have shown that the implications of trade aggregation for the first-order dynamics of NOEM models can be summarised by sufficient statistics that reflect the characteristics of the aggregator. These statistics include the steady-state values of consumption shares, bilateral elasticities and ratios related to the degree of homogeneity of the aggregator.

In a two-country model, the standard Armington aggregator is equivalent at first order to any other HOD(1) aggregator, with the same elasticity of substitution and consumption expenditure shares in steady state. This is also true in the case of HOD(h) aggregators, using our generalised HOD(h) Armington aggregator.

With $N > 2$ countries, the Armington aggregator can become restrictive to the extent that it imposes that the bilateral elasticities of substitution of each pair of goods are given by the same parameter. Other aggregators that allow these elasticities to be different in steady state can therefore affect the first-order dynamics of the model. This implies that the channel through which these aggregators affect the model is captured entirely by the asymmetries in the steady-state pairwise elasticities of substitution. Similarly, when compared to an alternative aggregator that is HOD(h), our generalised Armington aggregator can replicate the first-order dynamics under symmetry, but not under asymmetry.

Overall, our results guide precisely how alternative aggregators will impact dynamics compared to CES: specifically by changing the steady-state quantities of the sufficient statistics.

For clarity, the model we laid out was a simple endowment economy. However, as discussed, our results would continue to hold in more general settings—including cases in which trade occurs in other types of goods (e.g., intermediate inputs or investment goods), so long as the optimal composition of these goods, between domestic and foreign goods, remains an intratemporal problem. Nevertheless, a key assumption for our results is the separability of intertemporal and intratemporal decisions. While our results motivate further work assessing interactions between the two (e.g., Drozd, Kolbin, and Nosal, 2021), this assumption is common across most NOEM models. Finally, we derived all of these results analytically in a linearised model, and so they hold exactly at first order. We leave it for future research to explore the impact of the trade aggregator choice in different settings in which non-linearities may matter more.

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Appendix

A Proof of Theorem 1

For each country n, the relevant system of equations that are affected by the aggregator function, f, and the individual consumption levels, $\mathbf{c}_t^{(n)}$, are the definition of aggregate consumption, the definition of the price index, the $N-1$ relative demand functions, and the goods market clearing condition:

$$\begin{aligned}
C^{(n)} &= f(\mathbf{c}^{(n)}) \\
C^{(n)} &= \sum_{i=1}^{N} p_i^{(n)} c_i^{(n)} \\
\frac{p_i^{(n)}}{p_N^{(n)}} &= \frac{f_i^{(n)}}{f_N^{(n)}} \quad \forall\, i = 1, \dots N-1 \\
Y^{(n)} &= \sum_{i=1}^{N} c_n^{(i)}
\end{aligned}$$

where we have dropped the time subscripts for simplicity. In what follows, we will also drop the country (n) superscripts for simplicity, except for the derivations related to the goods market clearing condition where they are relevant.

We want to derive the log-linear form of these equations to understand what drives the first-order dynamics, and in particular how it depends on the function f. To do this, we will apply the general formula for the first-order Taylor expansion, and write each equation in a generic format $F(\mathbf{x}) = 0$, where $\mathbf{x}$ is the vector of all model variables. Then the multivariate first-order Taylor expansion around a point $\overline{\mathbf{x}}$ is given by:

$$\begin{aligned}
F(\mathbf{x}) &\approx \left(F'(\mathbf{x})\,|_{\mathbf{x}=\overline{\mathbf{x}}}\right)' (\mathbf{x} - \overline{\mathbf{x}}) \\
&= \sum_i \left.\frac{\partial F(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\overline{\mathbf{x}}} (x_i - \overline{x}_i) \\
&= \sum_i \overline{F}_i \overline{x}_i \widetilde{x}_i
\end{aligned}$$

where we use the notation $\widetilde{x} \equiv (x - \overline{x})/\overline{x}$, where $\overline{x}$ denotes the steady state, and $\overline{F}_i$ denotes the partial derivative $F_i(\overline{\mathbf{x}})$. We will apply this formula to each of the equations above.

Aggregate Consumption

$$\begin{aligned} C &= f(\mathbf{c}) \\ 0 &= C - f(\mathbf{c}) \\ &\approx \overline{C}\widetilde{C} - \sum_{i=1}^{N} \overline{f}_i \overline{c}_i \widetilde{c}_i \\ &= \widetilde{C} - \sum_{i=1}^{N} \frac{\overline{f}_i \overline{c}_i}{\overline{C}} \widetilde{c}_i \\ \widetilde{C} &\approx \sum_{i=1}^{N} \frac{\overline{f}_i \overline{c}_i}{\overline{C}} \widetilde{c}_i \end{aligned}$$

To simplify this equation, recall the FOCs of the cost-minimisation problem defined above:

$$Pp_i = \lambda f_i \quad \forall\, i = 1, \dots N$$

We can solve for the Lagrange multiplier using the definition of the aggregate price index:

$$\begin{aligned} C &= \sum_{i=1}^{N} p_i c_i \\ &= \sum_{i=1}^{N} \frac{\lambda f_i}{P} c_i \\ &= \frac{\lambda}{P} \sum_{i=1}^{N} f_i c_i \\ \lambda &= \frac{P}{\mathcal{H}(\mathbf{c})} \end{aligned}$$

where

$$\mathcal{H}(\mathbf{c}) = \frac{\sum_{i=1}^{N} f_i c_i}{C} = \frac{\sum_{i=1}^{N} f_i c_i}{f(\mathbf{c})}$$

Plugging this into the FOCs:

$$Pp_i = \frac{P}{\mathcal{H}(\mathbf{c})} f_i$$

or

$$p_i = \frac{f_i}{\mathcal{H}(\mathbf{c})} \qquad \Rightarrow \qquad f_i = p_i \mathcal{H}(\mathbf{c})$$

Hence

$$\frac{\overline{f}_i \overline{c}_i}{\overline{C}} = \mathcal{H}(\overline{\mathbf{c}}) \frac{\overline{p}_i \overline{c}_i}{\overline{C}}$$

Define the steady-state share of consumption expenditure on good i:

$$\overline{\alpha}_i \equiv \frac{\overline{p}_i \overline{c}_i}{\overline{C}}$$

Putting these together, denoting $\mathcal{H}(\overline{\mathbf{c}}) \equiv \overline{\mathcal{H}}$, the linearised form of the aggregator is given by:

$$\widetilde{C} \approx \overline{\mathcal{H}} \sum_{i=1}^{N} \overline{\alpha}_i \widetilde{c}_i$$

This depends on $\overline{\mathcal{H}}$ and $\overline{\alpha}_i$ for $i = 1, \ldots N$.

Consumer Price Index

$$\begin{aligned}
C &= \sum_{i=1}^{N} p_i c_i \\
0 &= C - \sum_{i=1}^{N} p_i c_i \\
&\approx \overline{C}\widetilde{C} - \sum_{i=1}^{N} \overline{p}_i \overline{c}_i \widetilde{c}_i - \sum_{i=1}^{N} \overline{c}_i \overline{p}_i \widetilde{p}_i \\
&\approx \widetilde{C} - \sum_{i=1}^{N} \frac{\overline{p}_i \overline{c}_i}{\overline{C}} \widetilde{c}_i - \sum_{i=1}^{N} \frac{\overline{p}_i \overline{c}_i}{\overline{C}} \widetilde{p}_i \\
&\approx \widetilde{C} - \sum_{i=1}^{N} \overline{\alpha}_i \widetilde{c}_i - \sum_{i=1}^{N} \overline{\alpha}_i \widetilde{p}_i
\end{aligned}$$

Again, this depends only on $\overline{\alpha}_i$ for $i = 1, \ldots N$.

Relative Demand Functions

Consider a specific i without loss of generality:

$$\begin{aligned}
\frac{p_i}{p_N} &= \frac{f_i(\mathbf{c})}{f_N(\mathbf{c})} \\
0 &= \frac{p_i}{p_N} - \frac{f_i(\mathbf{c})}{f_N(\mathbf{c})} \\
0 &\approx \frac{1}{\overline{p}_N} \overline{p}_i \widetilde{p}_i - \frac{\overline{p}_i}{\overline{p}_N^2} \overline{p}_N \widetilde{p}_N - \sum_{k=1}^{N} \left. \frac{\partial \left(\frac{f_i}{f_N} \right)}{\partial c_k} \right|_{ss} \overline{c}_k \widetilde{c}_k \\
&= \frac{\overline{p}_i}{\overline{p}_N} \left(\widetilde{p}_i - \widetilde{p}_N \right) - \sum_{k=1}^{N} \left. \frac{\partial \left(\frac{f_i}{f_N} \right)}{\partial c_k} \right|_{ss} \overline{c}_k \widetilde{c}_k
\end{aligned}$$

Consider the partial derivative term:

$$\begin{aligned}\frac{\partial\left(\frac{f_i}{f_N}\right)}{\partial c_k} &= \frac{1}{f_N}\frac{\partial f_i}{\partial c_k} - \frac{f_i}{f_N^2}\frac{\partial f_N}{\partial c_k} \\ &= \frac{f_{ik}}{f_N} - \frac{f_i f_{Nk}}{f_N^2} \\ &= \frac{f_i}{f_N}\left(\frac{f_{ik}}{f_i} - \frac{f_{Nk}}{f_N}\right)\end{aligned}$$

Plugging this back in:

$$\begin{aligned}0 &\approx \frac{\overline{p}_i}{\overline{p}_N}\left(\widetilde{p}_i - \widetilde{p}_N\right) - \sum_{k=1}^{N}\frac{f_i}{f_N}\left(\frac{f_{ik}}{f_i} - \frac{f_{Nk}}{f_N}\right)\bigg|_{ss}\overline{c}_k\widetilde{c}_k \\ &= \frac{\overline{p}_i}{\overline{p}_N}\left(\widetilde{p}_i - \widetilde{p}_N\right) - \frac{\overline{f}_i}{\overline{f}_N}\sum_{k=1}^{N}\left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k\widetilde{c}_k\end{aligned}$$

Using the fact that $\overline{p}_i/\overline{p}_N = \overline{f}_i/\overline{f}_N$:

$$\widetilde{p}_i - \widetilde{p}_N \approx \sum_{k=1}^{N}\left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k\widetilde{c}_k = \sum_{k=1}^{N} coef_k^{(iN)}\widetilde{c}_k$$

where $coef_k^{(iN)} \equiv \left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k$.

Consider now the definition of the elasticity of substitution between two different goods x and y (we consider here the direct partial elasticity as defined by McFadden (1963) or Sato (1967)):

$$\Phi_{xy} = \frac{\partial\ln\left(c_x/c_y\right)}{\partial\ln\left(f_y/f_x\right)} = -\left(\frac{1}{c_x f_x} + \frac{1}{c_y f_y}\right)\left[\left(\frac{f_{xx}}{f_x^2} - \frac{f_{xy}}{f_x f_y}\right) + \left(\frac{f_{yy}}{f_y^2} - \frac{f_{xy}}{f_x f_y}\right)\right]^{-1}$$

In a first step, we derive some relationships between the coefficients of the linearised relative demand function, the steady state bilateral elasticities and the steady state consumption shares.

$$\begin{aligned}\overline{\Phi}_{iN}^{-1} &= -\left[\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2} - \frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right) + \left(\frac{\overline{f}_{NN}}{\overline{f}_N^2} - \frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right)\right]\left(\frac{1}{\overline{c}_i\overline{f}_i} + \frac{1}{\overline{c}_N\overline{f}_N}\right)^{-1} \\ &= -\left[\frac{1}{\overline{f}_i\overline{c}_i}coef_i^{(iN)} - \frac{1}{\overline{f}_N\overline{c}_N}coef_N^{(iN)}\right]\left(\frac{1}{\overline{c}_i\overline{f}_i} + \frac{1}{\overline{c}_N\overline{f}_N}\right)^{-1} \\ &= -\left[\frac{1}{\overline{f}_i\overline{c}_i}coef_i^{(iN)} - \frac{1}{\overline{f}_N\overline{c}_N}coef_N^{(iN)}\right]\frac{\overline{c}_i\overline{c}_N\overline{f}_i\overline{f}_N}{\overline{c}_N\overline{f}_N + \overline{c}_i\overline{f}_i} \\ &= -\frac{\overline{c}_N\overline{f}_N}{\overline{c}_i\overline{f}_i + \overline{c}_N\overline{f}_N}coef_i^{(iN)} + \frac{\overline{c}_i\overline{f}_i}{\overline{c}_i\overline{f}_i + \overline{f}_N\overline{c}_N}coef_N^{(iN)}\end{aligned}$$

Using the definition of the steady state expenditure shares:

$$\frac{\overline{c}_i\overline{f}_i}{\overline{c}_i\overline{f}_i+\overline{f}_N\overline{c}_N} = \frac{\overline{c}_i\frac{\overline{f}_i}{\overline{f}_N}}{\overline{c}_i\frac{\overline{f}_i}{\overline{f}_N}+\overline{c}_N} = \frac{\overline{c}_i\frac{\overline{p}_i}{\overline{p}_N}}{\overline{c}_i\frac{\overline{p}_i}{\overline{p}_N}+\overline{c}_N}$$

$$= \frac{\overline{c}_i\overline{p}_i}{\overline{c}_i\overline{p}_i+\overline{p}_N\overline{c}_N} = \frac{\overline{c}_i\overline{p}_i}{\sum_l \overline{c}_l\overline{p}_l}\frac{\sum_l \overline{c}_l\overline{p}_l}{\overline{c}_i\overline{p}_i+\overline{p}_N\overline{c}_N}$$

$$= \frac{\overline{\alpha}_i}{\overline{\alpha}_i+\overline{\alpha}_N}$$

And we obtain:

$$\overline{\Phi}_{iN}^{-1} = -\frac{\overline{\alpha}_N}{\overline{\alpha}_i+\overline{\alpha}_N}coef_i^{(iN)} + \frac{\overline{\alpha}_i}{\overline{\alpha}_i+\overline{\alpha}_N}coef_N^{(iN)}$$

i.e.

$$(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\Phi}_{iN}^{-1} = -\overline{\alpha}_N coef_i^{(iN)} + \overline{\alpha}_i coef_N^{(iN)} \tag{A.1}$$

Equation (A.1) is the first type of relationship we were aiming for, and is true for every $i = 1, 2, ..., N-1$. Now we derive a second type of relationship, involving two bilateral elasticities.

$$\left(\frac{1}{\overline{c}_i\overline{f}_i}+\frac{1}{\overline{c}_k\overline{f}_k}\right)\overline{\Phi}_{ik}^{-1} - \left(\frac{1}{\overline{c}_N\overline{f}_N}+\frac{1}{\overline{c}_k\overline{f}_k}\right)\overline{\Phi}_{Nk}^{-1}$$

$$= -\left[\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)+\left(\frac{\overline{f}_{kk}}{\overline{f}_k^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)\right]$$

$$+\left[\left(\frac{\overline{f}_{NN}}{\overline{f}_N^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)+\left(\frac{\overline{f}_{kk}}{\overline{f}_k^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)\right]$$

$$= -\frac{1}{\overline{f}_i}\left(\frac{\overline{f}_{ii}}{\overline{f}_i}-\frac{\overline{f}_{ik}}{\overline{f}_k}\right)-\frac{1}{\overline{f}_k}\left(\frac{\overline{f}_{kk}}{\overline{f}_k}-\frac{\overline{f}_{ik}}{\overline{f}_i}\right)$$

$$+\frac{1}{\overline{f}_N}\left(\frac{\overline{f}_{NN}}{\overline{f}_N}-\frac{\overline{f}_{Nk}}{\overline{f}_k}\right)+\frac{1}{\overline{f}_k}\left(\frac{\overline{f}_{kk}}{\overline{f}_k}-\frac{\overline{f}_{Nk}}{\overline{f}_N}\right)$$

$$= -\frac{1}{\overline{f}_i}\left(\frac{\overline{f}_{ii}}{\overline{f}_i}-\frac{\overline{f}_{ik}}{\overline{f}_k}\right)+\frac{1}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}+\frac{1}{\overline{f}_N}\left(\frac{\overline{f}_{NN}}{\overline{f}_N}-\frac{\overline{f}_{Nk}}{\overline{f}_k}\right)$$

$$= -\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{ik}}{\overline{f}_i\overline{f}_k}\right)+\frac{1}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}+\left(\frac{\overline{f}_{NN}}{\overline{f}_N^2}-\frac{\overline{f}_{Nk}}{\overline{f}_N\overline{f}_k}\right)$$

$$= -\frac{\overline{f}_{ii}}{\overline{f}_i^2}+\frac{1}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}+\frac{\overline{f}_{NN}}{\overline{f}_N^2}+\frac{1}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}$$

$$= -\left(\frac{\overline{f}_{ii}}{\overline{f}_i^2}-\frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}\right)-\left(\frac{\overline{f}_{iN}}{\overline{f}_i\overline{f}_N}-\frac{\overline{f}_{NN}}{\overline{f}_N^2}\right)+\frac{2}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}$$

$$= -\frac{1}{\overline{c}_i\overline{f}_i}coef_i^{(iN)}-\frac{1}{\overline{c}_N\overline{f}_N}coef_N^{(iN)}+\frac{2}{\overline{c}_k\overline{f}_k}coef_k^{(iN)}$$

Now bringing back expenditure shares as above:

$$\left(\frac{\sum_l \overline{c}_l \overline{f}_l}{\overline{c}_i \overline{f}_i} + \frac{\sum_l \overline{c}_l \overline{f}_l}{\overline{c}_k \overline{f}_k}\right) \overline{\Phi}_{ik}^{-1} - \left(\frac{\sum_l \overline{c}_l \overline{f}_l}{\overline{c}_N \overline{f}_N} + \frac{\sum_l \overline{c}_l \overline{f}_l}{\overline{c}_k \overline{f}_k}\right) \overline{\Phi}_{Nk}^{-1}$$
$$= -\frac{\sum_l \overline{c}_l \overline{f}_l}{\overline{c}_i \overline{f}_i} coef_i^{(iN)} - \frac{\sum_l \overline{c}_l \overline{f}_l}{\overline{c}_N \overline{f}_N} coef_N^{(iN)} + \frac{2\sum_l \overline{c}_l \overline{f}_l}{\overline{c}_k \overline{f}_k} coef_k^{(iN)}$$

$$\left(\frac{1}{\overline{\alpha}_i} + \frac{1}{\overline{\alpha}_k}\right) \overline{\Phi}_{ik}^{-1} - \left(\frac{1}{\overline{\alpha}_N} + \frac{1}{\overline{\alpha}_k}\right) \overline{\Phi}_{Nk}^{-1}$$
$$= -\frac{1}{\overline{\alpha}_i} coef_i^{(iN)} - \frac{1}{\overline{\alpha}_N} coef_N^{(iN)} + \frac{2}{\overline{\alpha}_k} coef_k^{(iN)}$$
$$(\overline{\alpha}_i + \overline{\alpha}_k)\overline{\alpha}_N \overline{\Phi}_{ik}^{-1} - (\overline{\alpha}_k + \overline{\alpha}_N)\overline{\alpha}_i \overline{\Phi}_{Nk}^{-1}$$
$$= -\overline{\alpha}_k \overline{\alpha}_N coef_i^{(iN)} - \overline{\alpha}_i \overline{\alpha}_k coef_N^{(iN)} + 2\overline{\alpha}_i \overline{\alpha}_N coef_k^{(iN)} \tag{A.2}$$

Equation (A.2) is our second type of relationship, and is valid for all $i = 1, 2, ..., N-1$ and for all $k \neq i, N$. Now, we can use the relationships obtained in equations (A.1) and (A.2) to express the linearised relative demand function as a function of the steady state expenditure shares, elasticities and ratios $\mathcal{H}$ and $\mathcal{H}_j$.

From equation (A.1), we have:

$$coef_N^{(iN)} = \frac{\overline{\alpha}_i + \overline{\alpha}_N}{\overline{\alpha}_i} \overline{\Phi}_{iN}^{-1} + \frac{\overline{\alpha}_N}{\overline{\alpha}_i} coef_i^{(iN)} \tag{A.3}$$

And from equation (A.2), for all $k \neq i, N$:

$$\begin{aligned} coef_k^{(iN)} =& \frac{\overline{\alpha}_i + \overline{\alpha}_k}{2\overline{\alpha}_i} \overline{\Phi}_{ik}^{-1} - \frac{\overline{\alpha}_k + \overline{\alpha}_N}{2\overline{\alpha}_N} \overline{\Phi}_{Nk}^{-1} + \frac{\overline{\alpha}_k}{2\overline{\alpha}_i} coef_i^{(iN)} + \frac{\overline{\alpha}_k}{2\overline{\alpha}_N} coef_N^{(iN)} \\ =& \frac{\overline{\alpha}_i + \overline{\alpha}_k}{2\overline{\alpha}_i} \overline{\Phi}_{ik}^{-1} - \frac{\overline{\alpha}_k + \overline{\alpha}_N}{2\overline{\alpha}_N} \overline{\Phi}_{Nk}^{-1} + \frac{\overline{\alpha}_k}{2\overline{\alpha}_i} coef_i^{(iN)} \\ &+ \frac{\overline{\alpha}_k}{2\overline{\alpha}_N} \left(\frac{\overline{\alpha}_i + \overline{\alpha}_N}{\overline{\alpha}_i} \overline{\Phi}_{iN}^{-1} + \frac{\overline{\alpha}_N}{\overline{\alpha}_i} coef_i^{(iN)}\right) \\ =& \frac{\overline{\alpha}_k}{\overline{\alpha}_i} coef_i^{(iN)} + \frac{\overline{\alpha}_i + \overline{\alpha}_k}{2\overline{\alpha}_i} \overline{\Phi}_{ik}^{-1} - \frac{\overline{\alpha}_k + \overline{\alpha}_N}{2\overline{\alpha}_N} \overline{\Phi}_{Nk}^{-1} + \frac{(\overline{\alpha}_i + \overline{\alpha}_N)\overline{\alpha}_k}{2\overline{\alpha}_N \overline{\alpha}_i} \overline{\Phi}_{iN}^{-1} \end{aligned} \tag{A.4}$$

Plugging expressions (A.3) and (A.4) in the linearised relative demand function:

$$\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N =& coef_i^{(iN)}\widetilde{c}_i + \sum_{k=1,k\neq i}^{N-1} coef_k^{(iN)}\widetilde{c}_k + coef_N^{(iN)}\widetilde{c}_N \\
=& coef_i^{(iN)}\widetilde{c}_i \\
&+ \sum_{k=1,k\neq i}^{N-1}\left(\frac{\overline{\alpha}_k}{\overline{\alpha}_i}coef_i^{(iN)} + \frac{\overline{\alpha}_i+\overline{\alpha}_k}{2\overline{\alpha}_i}\overline{\Phi}_{ik}^{-1} - \frac{\overline{\alpha}_k+\overline{\alpha}_N}{2\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{2\overline{\alpha}_N\overline{\alpha}_i}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ \left(\frac{\overline{\alpha}_i+\overline{\alpha}_N}{\overline{\alpha}_i}\overline{\Phi}_{iN}^{-1} + \frac{\overline{\alpha}_N}{\overline{\alpha}_i}coef_i^{(iN)}\right)\widetilde{c}_N \\
=& coef_i^{(iN)}\widetilde{c}_i + \frac{\overline{\alpha}_N}{\overline{\alpha}_i}coef_i^{(iN)}\widetilde{c}_N + \sum_{k=1,k\neq i}^{N-1}\left(\frac{\overline{\alpha}_k}{\overline{\alpha}_i}coef_i^{(iN)}\right)\widetilde{c}_k \\
&+ \sum_{k=1,k\neq i}^{N-1}\left(\frac{\overline{\alpha}_i+\overline{\alpha}_k}{2\overline{\alpha}_i}\overline{\Phi}_{ik}^{-1} - \frac{\overline{\alpha}_k+\overline{\alpha}_N}{2\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{2\overline{\alpha}_N\overline{\alpha}_i}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ \frac{\overline{\alpha}_i+\overline{\alpha}_N}{\overline{\alpha}_i}\overline{\Phi}_{iN}^{-1}\widetilde{c}_N \\
&= \frac{1}{\overline{\alpha}_i}coef_i^{(iN)}\left(\overline{\alpha}_i\widetilde{c}_i + \overline{\alpha}_N\widetilde{c}_N + \sum_{k=1,k\neq i}^{N-1}\overline{\alpha}_k\widetilde{c}_k\right) \\
&+ \sum_{k=1,k\neq i}^{N-1}\left(\frac{\overline{\alpha}_i+\overline{\alpha}_k}{2\overline{\alpha}_i}\overline{\Phi}_{ik}^{-1} - \frac{\overline{\alpha}_k+\overline{\alpha}_N}{2\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{2\overline{\alpha}_N\overline{\alpha}_i}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ \frac{\overline{\alpha}_i+\overline{\alpha}_N}{\overline{\alpha}_i}\overline{\Phi}_{iN}^{-1}\widetilde{c}_N \\
=& \frac{1}{\overline{\alpha}_i}\Bigg[coef_i^{(iN)}\left(\sum_{k=1}^{N}\overline{\alpha}_k\widetilde{c}_k\right) \\
&+ \frac{1}{2}\sum_{k=1,k\neq i}^{N-1}\left((\overline{\alpha}_i+\overline{\alpha}_k)\,\overline{\Phi}_{ik}^{-1} - \frac{(\overline{\alpha}_k+\overline{\alpha}_N)\alpha_i}{\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{\overline{\alpha}_N}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ (\overline{\alpha}_i+\overline{\alpha}_N)\overline{\Phi}_{iN}^{-1}\widetilde{c}_N\Bigg]
\end{aligned}$$

From the aggregate consumption linearisation we know that:

$$\widetilde{C} \approx \overline{\mathcal{H}}\sum_{k=1}^{N}\overline{\alpha}_k\widetilde{c}_k$$

So we get:

$$\begin{aligned}
\overline{\alpha}_i\left(\widetilde{p}_i - \widetilde{p}_N\right) =& coef_i^{(iN)}\frac{\widetilde{C}}{\overline{\mathcal{H}}} \\
&+ \frac{1}{2}\sum_{k=1,k\neq i}^{N-1}\left((\overline{\alpha}_i+\overline{\alpha}_k)\,\overline{\Phi}_{ik}^{-1} - \frac{(\overline{\alpha}_k+\overline{\alpha}_N)\overline{\alpha}_i}{\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1} + \frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{\overline{\alpha}_N}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_k \\
&+ (\overline{\alpha}_i+\overline{\alpha}_N)\overline{\Phi}_{iN}^{-1}\widetilde{c}_N
\end{aligned} \tag{A.5}$$

With a similar approach, still using equations (A.1) and (A.2), we can obtain the following expressions for the coefficients and the linearised relative demand function:

$$
\begin{aligned}
coef_i^{(iN)} &= \frac{\overline{\alpha}_i}{\alpha_N} coef_N^{(iN)} - \frac{\overline{\alpha}_i + \overline{\alpha}_N}{\overline{\alpha}_N} \overline{\Phi}_{iN}^{-1} \\
coef_k^{(iN)} &= \frac{\overline{\alpha}_k}{\overline{\alpha}_N} coef_N^{(iN)} + \frac{\overline{\alpha}_i + \overline{\alpha}_k}{2\overline{\alpha}_i} \overline{\Phi}_{ik}^{-1} - \frac{\overline{\alpha}_k + \overline{\alpha}_N}{2\overline{\alpha}_N} \overline{\Phi}_{Nk}^{-1} - \frac{\overline{\alpha}_k(\overline{\alpha}_i + \overline{\alpha}_N)}{2\overline{\alpha}_i\overline{\alpha}_N} \overline{\Phi}_{iN}^{-1} \quad \forall k \neq i, N
\end{aligned}
$$

implying:

$$
\begin{aligned}
\overline{\alpha}_N \left(\widetilde{p}_i - \widetilde{p}_N\right) = & \, coef_N^{(iN)} \frac{\widetilde{C}}{\overline{\overline{\mathcal{H}}}} \\
& + \frac{1}{2} \sum_{k=1, k\neq i}^{N-1} \left(\frac{(\overline{\alpha}_i + \overline{\alpha}_k)\overline{\alpha}_N}{\overline{\alpha}_i} \overline{\Phi}_{ik}^{-1} - \left(\overline{\alpha}_k + \overline{\alpha}_N\right) \overline{\Phi}_{Nk}^{-1} - \frac{\overline{\alpha}_k(\overline{\alpha}_i + \overline{\alpha}_N)}{\overline{\alpha}_i} \overline{\Phi}_{iN}^{-1} \right) \widetilde{c}_k \\
& - (\overline{\alpha}_i + \overline{\alpha}_N) \overline{\Phi}_{iN}^{-1} \widetilde{c}_i
\end{aligned} \tag{A.6}
$$

Using again a similar approach, from equations (A.1) and (A.2):

$$
\begin{aligned}
coef_N^{(iN)} =& \frac{\overline{\alpha}_k + \overline{\alpha}_N}{2\overline{\alpha}_k} \overline{\Phi}_{Nk}^{-1} - \frac{(\overline{\alpha}_i + \overline{\alpha}_k)\overline{\alpha}_N}{2\overline{\alpha}_i\overline{\alpha}_k} \overline{\Phi}_{ik}^{-1} + \frac{\overline{\alpha}_i + \overline{\alpha}_N}{2\overline{\alpha}_i} \overline{\Phi}_{iN}^{-1} \\
& + \frac{\overline{\alpha}_N}{\overline{\alpha}_k} coef_k^{(iN)} \quad \forall k \neq i, N
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
coef_i^{(iN)} =& \frac{\overline{\alpha}_i}{\alpha_N} coef_N^{(iN)} - \frac{\overline{\alpha}_i + \overline{\alpha}_N}{\alpha_N} \overline{\Phi}_{iN}^{-1} \\
=& \frac{\overline{\alpha}_i}{\overline{\alpha}_k} coef_k^{(iN)} + \frac{\overline{\alpha}_i(\overline{\alpha}_k + \overline{\alpha}_N)}{2\overline{\alpha}_k\overline{\alpha}_N} \overline{\Phi}_{Nk}^{-1} - \frac{\overline{\alpha}_i + \overline{\alpha}_k}{2\overline{\alpha}_k} \overline{\Phi}_{ik}^{-1} \\
& - \frac{\overline{\alpha}_i + \overline{\alpha}_N}{2\overline{\alpha}_N} \overline{\Phi}_{iN}^{-1} \quad \forall k \neq i, N
\end{aligned} \tag{A.8}
$$

Considering a specific $k \neq i, N$ without loss of generality, we now need to also express $coef_l^{(iN)}$ $(l \neq i, N, k)$ as a function of $coef_k^{(iN)}$, steady state bilateral elasticities and consumption shares. Rewriting equation (A.7) for any $l \neq i, N, k$:

$$
coef_N^{(iN)} = \frac{\overline{\alpha}_l + \overline{\alpha}_N}{2\overline{\alpha}_l} \overline{\Phi}_{Nl}^{-1} - \frac{(\overline{\alpha}_i + \overline{\alpha}_l)\overline{\alpha}_N}{2\overline{\alpha}_i\overline{\alpha}_l} \overline{\Phi}_{il}^{-1} + \frac{\overline{\alpha}_i + \overline{\alpha}_N}{2\overline{\alpha}_i} \overline{\Phi}_{iN}^{-1} + \frac{\overline{\alpha}_N}{\overline{\alpha}_l} coef_l^{(iN)}
$$

Implying for all $l \neq i, N, k$:

$$
\begin{aligned}
coef_l^{(iN)} =& \frac{\overline{\alpha}_l}{\overline{\alpha}_N} \left(coef_N^{(iN)} + \frac{(\overline{\alpha}_i + \overline{\alpha}_l)\overline{\alpha}_N}{2\overline{\alpha}_i\overline{\alpha}_l} \overline{\Phi}_{il}^{-1} - \frac{\overline{\alpha}_l + \overline{\alpha}_N}{2\overline{\alpha}_l} \overline{\Phi}_{Nl}^{-1} - \frac{\overline{\alpha}_i + \overline{\alpha}_N}{2\overline{\alpha}_i} \overline{\Phi}_{iN}^{-1} \right) \\
=& \frac{\overline{\alpha}_l}{\overline{\alpha}_k} coef_k^{(iN)} + \frac{\overline{\alpha}_l(\overline{\alpha}_k + \overline{\alpha}_N)}{2\overline{\alpha}_k\overline{\alpha}_N} \overline{\Phi}_{Nk}^{-1} \\
& - \frac{\overline{\alpha}_l(\overline{\alpha}_i + \overline{\alpha}_k)}{2\overline{\alpha}_i\overline{\alpha}_k} \overline{\Phi}_{ik}^{-1} + \frac{(\overline{\alpha}_i + \overline{\alpha}_l)}{2\overline{\alpha}_i} \overline{\Phi}_{il}^{-1} - \frac{\overline{\alpha}_l + \overline{\alpha}_N}{2\overline{\alpha}_N} \overline{\Phi}_{Nl}^{-1}
\end{aligned} \tag{A.9}
$$

We can again plug the expressions (A.7) to (A.9) into the linearised relative demand function and obtain after some manipulations:

$$\begin{aligned}
\overline{\alpha}_k\left(\widetilde{p}_i-\widetilde{p}_N\right) =& coef_k^{(iN)}\frac{\widetilde{C}}{\overline{\overline{\mathcal{H}}}} \\
&+\frac{1}{2}\left(\frac{\overline{\alpha}_i(\overline{\alpha}_k+\overline{\alpha}_N)}{\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1}-\left(\overline{\alpha}_i+\overline{\alpha}_k\right)\overline{\Phi}_{ik}^{-1}-\frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{\overline{\alpha}_N}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_i \\
&+\frac{1}{2}\left(\left(\overline{\alpha}_k+\overline{\alpha}_N\right)\overline{\Phi}_{Nk}^{-1}-\frac{(\overline{\alpha}_i+\overline{\alpha}_k)\overline{\alpha}_N}{\overline{\alpha}_i}\overline{\Phi}_{ik}^{-1}+\frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{\overline{\alpha}_i}\overline{\Phi}_{iN}^{-1}\right)\widetilde{c}_N \\
&+\frac{1}{2}\sum_{l=1,l\neq i,k}^{N-1}\left(\frac{\overline{\alpha}_l(\overline{\alpha}_k+\overline{\alpha}_N)}{\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1}\frac{\overline{\alpha}_l(\overline{\alpha}_i+\overline{\alpha}_k)}{\overline{\alpha}_i}\overline{\Phi}_{ik}^{-1}\right. \\
&\left.+\frac{(\overline{\alpha}_i+\overline{\alpha}_l)\overline{\alpha}_k}{\overline{\alpha}_i}\overline{\Phi}_{il}^{-1}-\frac{(\overline{\alpha}_l+\overline{\alpha}_N)\overline{\alpha}_k}{\overline{\alpha}_N}\overline{\Phi}_{Nl}^{-1}\right)\widetilde{c}_l
\end{aligned} \tag{A.10}$$

Equation (A.10) is valid for any $k \neq i, N$. Now let's sum equations (A.5), (A.6) and all (A.10) for all $k \neq i, N$ and notice that by definition of the consumption shares:

$$\overline{\alpha}_i\left(\widetilde{p}_i-\widetilde{p}_N\right)+\overline{\alpha}_N\left(\widetilde{p}_i-\widetilde{p}_N\right)+\sum_{k=1,k\neq i}^{N-1}\overline{\alpha}_k\left(\widetilde{p}_i-\widetilde{p}_N\right)=\left(\widetilde{p}_i-\widetilde{p}_N\right)\sum_{k=1}^{N}\overline{\alpha}_k=\widetilde{p}_i-\widetilde{p}_N$$

And we obtain the following expression for the linearised relative demand function:

$$\begin{aligned}
\widetilde{p}_i-\widetilde{p}_N =& \frac{\widetilde{C}}{\overline{\overline{\mathcal{H}}}}\sum_{k=1}^{N}\left(coef_k^{(iN)}\right) \\
&+\widetilde{c}_i\left[\frac{1}{2}\sum_{k=1,k\neq i}^{N-1}\left(\frac{\overline{\alpha}_i(\overline{\alpha}_k+\overline{\alpha}_N)}{\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1}-\left(\overline{\alpha}_i+\overline{\alpha}_k\right)\overline{\Phi}_{ik}^{-1}-\frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{\overline{\alpha}_N}\overline{\Phi}_{iN}^{-1}\right)\right. \\
&\left.-\left(\overline{\alpha}_i+\overline{\alpha}_N\right)\overline{\Phi}_{iN}^{-1}\right] \\
&+\widetilde{c}_N\left[\frac{1}{2}\sum_{k=1,k\neq i}^{N-1}\left(\left(\overline{\alpha}_k+\overline{\alpha}_N\right)\overline{\Phi}_{Nk}^{-1}-\frac{(\overline{\alpha}_i+\overline{\alpha}_k)\overline{\alpha}_N}{\overline{\alpha}_i}\overline{\Phi}_{ik}^{-1}+\frac{(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k}{\overline{\alpha}_i}\overline{\Phi}_{iN}^{-1}\right)\right. \\
&\left.+\left(\overline{\alpha}_i+\overline{\alpha}_N\right)\overline{\Phi}_{iN}^{-1}\right] \\
&+\frac{1}{2}\sum_{k=1,k\neq i}^{N-1}\widetilde{c}_k\left(\left(\overline{\alpha}_i+\overline{\alpha}_k\right)\left(1+\frac{\overline{\alpha}_N}{\overline{\alpha}_i}\right)\overline{\Phi}_{ik}^{-1}-\left(\overline{\alpha}_k+\overline{\alpha}_N\right)\left(\frac{\overline{\alpha}_i}{\overline{\alpha}_N}+1\right)\overline{\Phi}_{Nk}^{-1}\right. \\
&\left.+(\overline{\alpha}_i+\overline{\alpha}_N)\overline{\alpha}_k\left(\frac{1}{\overline{\alpha}_N}-\frac{1}{\overline{\alpha}_i}\right)\overline{\Phi}_{iN}^{-1}\right) \\
&+\frac{1}{2}\sum_{k=1,k\neq i}^{N-1}\left[\sum_{l=1,l\neq i,k}^{N-1}\widetilde{c}_l\left(\frac{\overline{\alpha}_l(\overline{\alpha}_k+\overline{\alpha}_N)}{\overline{\alpha}_N}\overline{\Phi}_{Nk}^{-1}-\frac{\overline{\alpha}_l(\overline{\alpha}_i+\overline{\alpha}_k)}{\overline{\alpha}_i}\overline{\Phi}_{ik}^{-1}+\frac{(\overline{\alpha}_i+\overline{\alpha}_l)\overline{\alpha}_k}{\overline{\alpha}_i}\overline{\Phi}_{il}^{-1}\right.\right. \\
&\left.\left.-\frac{(\overline{\alpha}_l+\overline{\alpha}_N)\overline{\alpha}_k}{\overline{\alpha}_N}\overline{\Phi}_{Nl}^{-1}\right)\right]
\end{aligned}$$

Despite a fairly rich expression, this expression depends only on steady state elasticities, consumption shares and the term $\frac{\widetilde{C}}{\overline{\mathcal{H}}}\sum_{k=1}^{N}\left(coef_k^{(iN)}\right)$. Note that:

$$\begin{aligned}
\sum_{k=1}^{N} coef_k^{(iN)} &= \sum_{k=1}^{N}\left(\frac{\overline{f}_{ik}}{\overline{f}_i} - \frac{\overline{f}_{Nk}}{\overline{f}_N}\right)\overline{c}_k \\
&= \sum_{k=1}^{N}\left(\frac{\overline{f}_{ik}\overline{c}_k}{\overline{f}_i}\right) - \sum_{k=1}^{N}\left(\frac{\overline{f}_{Nk}\overline{c}_k}{\overline{f}_N}\right) \\
&= \overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N
\end{aligned}$$

where $\overline{\mathcal{H}}_l \equiv \frac{\sum_{k=1}^{N}\overline{f}_{lk}\overline{c}_k}{\overline{f}_l}$ for all $l = 1, 2, ..., N$.

and therefore, after some rearranging:

$$\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N =& \widetilde{C}\frac{\overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N}{\overline{\mathcal{H}}} \\
&+ \frac{1}{2}\sum_{k=1}^{N}\widetilde{c}_k \sum_{l=1, l\neq k}^{N}\left(\overline{\alpha}_k\left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1}\right) + \overline{\alpha}_l\left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1}\right)\right. \\
&+ \left.\frac{\overline{\alpha}_k\overline{\alpha}_l}{\overline{\alpha}_N}\left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1}\right) + \frac{\overline{\alpha}_k\overline{\alpha}_l}{\overline{\alpha}_i}\left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1}\right)\right)
\end{aligned}$$

with the convention that $\Phi_{xy}^{-1} = 0$ if $y = x$.

Recalling that $\widetilde{C}$ can be expressed as a function of the consumptions $\widetilde{c}_l$, the steady state ratio $\overline{\mathcal{H}}$ and the steady state consumption shares, the equation above defines the linearised demand function as depending only on steady state consumption shares $\overline{\alpha}_l$, steady state bilateral elasticities $\overline{\Phi}_{lm}$ and the steady state ratios $\overline{\mathcal{H}}$ and $\overline{\mathcal{H}}_l$ ($l = 1, 2, ..., N$; $m = 1, 2, ..., N$).

Considering the above equation for all $i = 1, ... N-1$, we have prooved that the first-order dynamics of all relative demand functions depend only on the steady state values of the sufficient statistics listed in Theorem 1.

Market Clearing Condition

$$
\begin{aligned}
Y^{(n)} &= \sum_{i=1}^{N} c_n^{(i)} \\
0 &= Y^{(n)} - \sum_{i=1}^{N} c_n^{(i)} \\
&\approx \overline{Y}^{(n)} \widetilde{Y}^{(n)} - \sum_{i=1}^{N} \overline{c}_n^{(i)} \widetilde{c}_n^{(i)} \\
\widetilde{Y}^{(n)} &\approx \sum_{i=1}^{N} \frac{\overline{c}_n^{(i)}}{\overline{Y}^{(n)}} \widetilde{c}_n^{(i)}
\end{aligned}
$$

Recall that we assumed that in steady state $\overline{C}^{(n)} = \overline{p}_n^{(n)} \overline{Y}^{(n)}$, which implies:

$$
\widetilde{Y}^{(n)} \approx \sum_{i=1}^{N} \frac{\overline{p}_n^{(n)} \overline{c}_n^{(i)}}{\overline{C}^{(n)}} \widetilde{c}_n^{(i)}
$$

We further assume that there is bilaterally balanced trade between every country-pair at steady state, which means that $\overline{p}_i^{(n)} \overline{c}_i^{(n)} = \overline{p}_n^{(i)} \overline{c}_n^{(i)}$. Plugging this in, assuming that the law of one price holds ($\overline{p}_n^{(i)} = \overline{p}_n^{(n)}$):

$$
\begin{aligned}
\widetilde{Y}^{(n)} &\approx \sum_{i=1}^{N} \frac{\overline{p}_i^{(n)} \overline{c}_i^{(n)}}{\overline{C}^{(n)}} \widetilde{c}_n^{(i)} \\
&= \sum_{i=1}^{N} \overline{\alpha}_i^{(n)} \widetilde{c}_n^{(i)}
\end{aligned}
$$

which again only depends on the $\overline{\alpha}_i^{(n)}$. □

B Proof of Corollary 1

As shown in Appendix A, the linearised equations characterising the aggregate consumption, the consumer price index and the market clearing conditions already depend only on steady state consumption shares, steady state bilateral elasticities and the steady state ratio $\overline{\mathcal{H}}$. Recall now that the linearised relative demand function equations are defined for all $i = 1, 2, ..., N$ as:

$$
\begin{aligned}
\widetilde{p}_i - \widetilde{p}_N = &\widetilde{C} \frac{\overline{\mathcal{H}}_i - \overline{\mathcal{H}}_N}{\overline{\mathcal{H}}} \\
&+ \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k \sum_{l=1, l \neq k}^{N} \left(\overline{\alpha}_k \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1} \right) + \overline{\alpha}_l \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) \right. \\
&\left. + \frac{\overline{\alpha}_k \overline{\alpha}_l}{\overline{\alpha}_N} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) + \frac{\overline{\alpha}_k \overline{\alpha}_l}{\overline{\alpha}_i} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1} \right) \right)
\end{aligned}
$$

It is easy to check that a function f homogeneous of degree r has the following property:

$$\frac{\sum_{k=1}^{N} f_{ik} c_k}{f_i(\mathbf{c})} = r - 1$$

This implies that $\overline{\mathcal{H}}_i = r - 1$ for all $i = 1, 2, ..., N$. Hence the first term in the linearised relative demand functions is equal to zero, and the steady state consumption shares, bilateral elasticities and the ratio $\overline{\mathcal{H}}$ are sufficient to characterise the dynamics of the model at first order.

$$\begin{aligned}\widetilde{p}_i - \widetilde{p}_N =& \frac{1}{2} \sum_{k=1}^{N} \widetilde{c}_k \sum_{l=1, l\neq k}^{N} \left(\overline{\alpha}_k \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{il}^{-1} \right) + \overline{\alpha}_l \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) \right. \\ &+ \left. \frac{\overline{\alpha}_k \overline{\alpha}_l}{\overline{\alpha}_N} \left(\overline{\Phi}_{Nl}^{-1} - \overline{\Phi}_{Nk}^{-1} \right) + \frac{\overline{\alpha}_k \overline{\alpha}_l}{\overline{\alpha}_i} \left(\overline{\Phi}_{ik}^{-1} - \overline{\Phi}_{il}^{-1} \right) \right)\end{aligned}$$

□

C Kimball Aggregator Derivations

In this Appendix, we derive the elasticity of substitution between two goods i and j, for $i, j = 1, 2, ..., N$ and $i \neq j$, for a representative consumer in country n, where $n = 1, 2, ..., N$, implied by the Kimball (1995) aggregator. For readability, we drop the country (n) superscripts and the time subscripts. The elasticity of substitution that we derive is defined as:

$$\Phi_{ij} = \frac{\mathrm{d}(c_j/c_i)}{\mathrm{d}(p_i/p_j)} \frac{c_i p_i}{c_j p_j}$$

We note that the first term on the right-hand side of this expression can be written as:

$$\begin{aligned}\frac{\mathrm{d}(c_j/c_i)}{\mathrm{d}(p_i/p_j)} &= \left[\frac{\mathrm{d}(p_i/p_j)}{\mathrm{d}(c_j/c_i)} \right]^{-1} \\ &= \left[\frac{\mathrm{d}(p_i/p_j)}{\mathrm{d}c_j} \frac{\mathrm{d}c_j}{\mathrm{d}(c_j/c_i)} \right]^{-1} \end{aligned} \tag{C.1}$$

We derive this term in two steps.

First, we solve for the final term in equation (C.1), which can be expressed as:

$$\begin{aligned}\frac{\mathrm{d}c_j}{\mathrm{d}(c_j/c_i)} &= \left[\frac{\mathrm{d}(c_j/c_i)}{\mathrm{d}c_j} \right]^{-1} \\ &= \left[\frac{\partial(c_j/c_i)}{\partial c_j} + \frac{\partial(c_j/c_i)}{\partial c_i} \frac{\mathrm{d}c_i}{\mathrm{d}c_j} \right]^{-1}\end{aligned}$$

Within this, we can solve for $\frac{\mathrm{d}c_i}{\mathrm{d}c_j}$ by using the total derivative of the aggregator function $C = f(\mathbf{c})$, where $\mathrm{d}C = 0$ and $\mathrm{d}c_k = 0$ for all $k = 1, 2, ..., N$ where $k \neq i, j$. This yields:

$$\frac{\mathrm{d}c_i}{\mathrm{d}c_j} = -\frac{f_j}{f_i} = -\frac{p_j}{p_i}$$

So then:

$$\frac{\mathrm{d}c_j}{\mathrm{d}(c_j/c_i)} = \left[\frac{1}{c_i}\left(1 + \frac{p_j c_j}{p_i c_i}\right)\right]^{-1}$$

Second, we solve for the first term in equation (C.1). To do this, we use the fact that our specification of $\Upsilon(.)$ yields the following derivative:

$$\Upsilon'(x) = \frac{\sigma - 1}{\sigma} \exp\left\{\frac{1 - x^{\frac{\epsilon}{\sigma}}}{\epsilon}\right\}$$

and we note that the relative demand function can be expressed as:

$$\frac{p_i}{p_j} = \frac{\Upsilon'\left(\frac{c_i}{b_i C}\right)}{\Upsilon'\left(\frac{c_j}{b_j C}\right)} = \frac{\exp\left\{\frac{1}{\epsilon}\left(1 - \left(\frac{c_{i,t}^{(n)}}{b_i^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}}{\exp\left\{\frac{1}{\epsilon}\left(1 - \left(\frac{c_{j,t}^{(n)}}{b_j^{(n)} C_t^{(n)}}\right)^{\frac{\epsilon}{\sigma}}\right)\right\}} \equiv h\left(c_i, c_j, C\right)$$

So then, when $\mathrm{d}C = 0$:

$$\begin{aligned}
\frac{\mathrm{d}(p_i/p_j)}{\mathrm{d}c_j} =& \frac{\mathrm{d}h}{\mathrm{d}c_j} \\
=& \frac{\partial h}{\partial c_j} + \frac{\partial h}{\partial c_i}\frac{\mathrm{d}c_i}{\mathrm{d}c_j} \\
=& \frac{1}{\sigma c_i}\left(\frac{c_i}{b_i C}\right)^{\frac{\epsilon}{\sigma}} + \frac{p_i}{p_j}\frac{1}{\sigma c_j}\left(\frac{c_j}{b_j C}\right)^{\frac{\epsilon}{\sigma}}
\end{aligned}$$

Combining the expressions for the first and second terms in equation (C.1) yields:

$$\frac{\mathrm{d}(c_j/c_i)}{\mathrm{d}(p_i/p_j)} = \left(\frac{1}{\sigma c_i}\left(\frac{c_i}{b_i C}\right)^{\frac{\epsilon}{\sigma}} + \frac{p_i}{p_j}\frac{1}{\sigma c_j}\left(\frac{c_j}{b_j C}\right)^{\frac{\epsilon}{\sigma}}\right) c_i \left[1 + \frac{p_j c_j}{p_i c_i}\right]^{-1}$$

With this, the elasticity of substitution can be written as:

$$\Phi_{ij} = \sigma\left(1 + \frac{\alpha_i}{\alpha_j}\right)\left[\left(\frac{c_i}{b_i C}\right)^{\frac{\epsilon}{\sigma}} + \frac{\alpha_i}{\alpha_j}\left(\frac{c_j}{b_j C}\right)^{\frac{\epsilon}{\sigma}}\right]^{-1} \tag{C.2}$$

D Kimball Aggregator Properties

In this Appendix, we provide more detail on the properties of the Kimball aggregator in the two-country case.

To further explore the properties of the Kimball aggregator, Figure D.1 plots the relative-demand function, given by equation (8) in the main text, and corresponding elasticities from equation (10) in the main text, for different values of ϵ. To form this plot, we calibrate the three remaining aggregator parameters: $\sigma = 1.5$, $b_1^{(n)} = 0.8$ and $b_2^{(n)} = 0.2$. First, notice that when $p_1^{(n)}/p_2^{(n)} = 1$, we have $c_1^{(n)}/c_2^{(n)} = b_1^{(n)}/b_2^{(n)} = 4$ and $\Phi_{1,2}^{(n)} = \sigma = 1.5$ independently of ϵ. This implies that Kimball is also equivalent to CES at the point of symmetry across good types, with $\sigma = \phi$ and $b_1^{(n)} = a_1^{(n)}$.[13]

More generally, ϵ controls the curvature of the demand function. In the limiting case of CES preferences, as $\epsilon \to 0$, shown in the black dotted lines, the relative-demand function is convex and the elasticity of substitution is constant at $\sigma = 1.5$. As ϵ increases, the relative-demand curve becomes less convex, and the elasticity of substitution varies with the relative consumption levels. For $\epsilon = \sigma$, the relative-demand curve is approximately linear. When $\epsilon > \sigma$, the curve is concave. When this is the case, the concave relative-demand curves imply finite 'choke prices', above which demand for the relatively more expensive good is 0.

Consider, for example, the concave relative demand at $\epsilon = 5$ in panel (a) of Figure D.1. Here, as the price of good 1 relative to good 2 rises above 1, the concavity of the curve means that relative demand for good 1 falls more than in the CES case. In contrast, when the relative price of good 1 falls below 1, the concavity of the curve means that the relative demand for good 1 rises less rapidly than it does under CES.

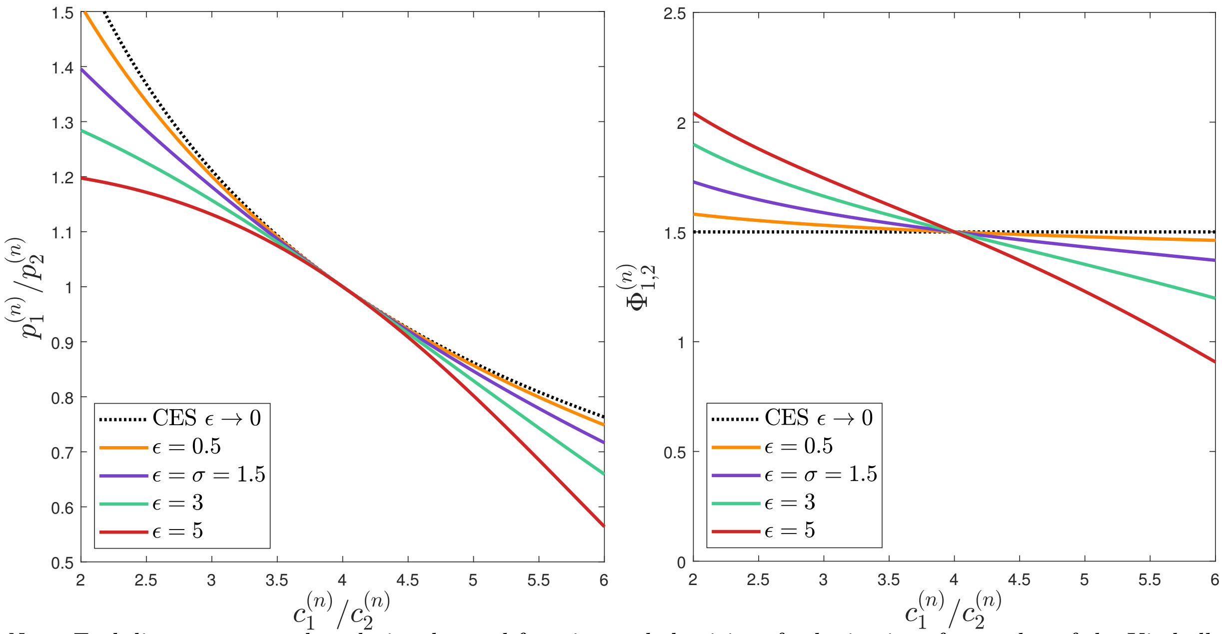
We can explain this equivalently in terms of the elasticity of substitution, shown in panel (b) of Figure D.1. When consumption of good 1 is low relative to good 2, then the elasticity of substitution rises, and a decrease in the relative price of good 1 leads to a larger substitution towards good 1. Conversely, when the consumption of good 1 is high relative to good 2, then the elasticity of substitution falls, and a decrease in the relative price of good 1 leads to a smaller substitution towards good 1.

This relative-demand curvature allows for the elasticity of substitution to vary over time, as the economy is hit by exogenous shocks, as illustrated by the time subscripts in equation (10) in the main draft. This leads to what Klenow and Willis (2016) refer to as "a smoothed version of a kinked demand curve": if a shock drives the relative price of a good up, the elasticity of substitution increases, such that demand declines more than the CES case, while if a shock drives the relative price down, the elasticity decreases, such that demand increases less than the CES case.

[13]This point is explored more in Baqaee, Farhi, and Sangani (2021), who highlight the importance of firm heterogeneity when using the Kimball aggregator to aggregate across monopolistically differentiated goods.

Figure D.1: Kimball (1995) aggregator

(a) Relative-demand functions (b) Elasticity of substitution



Note: Each line represents the relative-demand function and elasticity of substitutions for a value of the Kimball 'curvature' parameter ϵ, plotted with $\sigma = 1.5$ and $b_1^{(n)} = 0.8$.

E Nested-CES with $N > 2$ Countries

In section 4.1.2, we compared Kimball to a single-layer Armington aggregator, which implied by definition that the bilateral elasticities were the same across all country-good pairs. One way to gain flexibility, while retaining the tractability of the Armington aggregator, is to move to a nested-CES framework, i.e. a layered sequence of two-good CES aggregators that recursively adds each country-good to the current bundle. This will imply $(N-1)$ layers, which allows for $(N-1)$ elasticity parameters instead of a single one. However, this framework cannot replicate any alternative to the Armington aggregator, as we have now $(N-1)$ nested-CES elasticity parameters, but N bilateral elasticities to match for each country, and are therefore still missing one degree of freedom. We provide supporting derivations and a numerical example in Appendices E.1 and E.2, respectively.

E.1 Derivations with $N = 3$

Here we derive the elasticity of substitution between pairs of goods in the 3-country nested CES setup detailed in the main text. We consider country (H) along all computations here and therefore drop the superscripts (H) and the time subscripts for readability. Let us recall the formula for the direct partial elasticity between goods x and y:

$$\Phi_{xy}^{-1} = -\left(\frac{1}{c_x f_x} + \frac{1}{c_y f_y}\right)^{-1} \left[\left(\frac{f_{xx}}{f_x^2} - \frac{f_{xy}}{f_x f_y}\right) + \left(\frac{f_{yy}}{f_y^2} - \frac{f_{xy}}{f_x f_y}\right)\right]$$

We can apply it to the 3-country nested CES aggregator defined by :

$$C = f\left(c_H, c_F, c_R\right) = \left({a_H}^{\frac{1}{\phi_H}} {c_H}^{\frac{\phi_H - 1}{\phi_H}} + (1 - a_H)^{\frac{1}{\phi_H}} {C_{FR}}^{\frac{\phi_H - 1}{\phi_H}} \right)^{\frac{\phi_H}{\phi_H - 1}}$$

$$\text{where} \qquad C_{FR} = \left({a_F}^{\frac{1}{\phi_F}} {c_F}^{\frac{\phi_F - 1}{\phi_F}} + (1 - a_F)^{\frac{1}{\phi_F}} {C_R}^{\frac{\phi_F - 1}{\phi_F}} \right)^{\frac{\phi_F}{\phi_F - 1}}$$

First, we compute the partial derivatives of f.

$$\begin{aligned}
f_H &= a_H^{\frac{1}{\phi_H}} \left(\frac{C}{c_H}\right)^{\frac{1}{\phi_H}} \\
f_F &= (1 - a_H)^{\frac{1}{\phi_H}} a_F^{\frac{1}{\phi_F}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \\
f_R &= (1 - a_H)^{\frac{1}{\phi_H}} (1 - a_F)^{\frac{1}{\phi_F}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \\
f_{HH} &= \frac{1}{\phi_H} f_H \left(-\frac{1}{c_H} + \frac{1}{C} f_H\right) \\
f_{HF} = f_{FH} &= \frac{1}{\phi_H} a_H^{\frac{1}{\phi_H}} (1 - a_H)^{\frac{1}{\phi_H}} a_F^{\frac{1}{\phi_F}} \left(\frac{C}{c_H}\right)^{\frac{1}{\phi_H}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C} = \frac{1}{\phi_H C} f_H f_F \\
f_{HR} &= \frac{1}{\phi_H C} f_H f_R \\
f_{FF} &= f_F \left(-\frac{1}{\phi_F c_F} + \frac{\phi_H - \phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} f_F\right) \\
f_{FR} &= f_F (1 - a_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \left(\frac{\phi_H - \phi_F}{\phi_H \phi_F} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} (1 - a_H)^{\frac{1}{\phi_H}} \left(\frac{C}{C_{FR}}\right)^{\frac{1}{\phi_H}}\right) \\
&= f_F \left(\frac{1}{\phi_H C} f_R + \frac{\phi_H - \phi_F}{\phi_H \phi_F} \frac{1}{C_{FR}} (1 - \alpha_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}}\right) \\
f_{RR} &= f_R \left(-\frac{1}{\phi_F c_R} + \frac{\phi_H - \phi_F}{\phi_H \phi_F} (1 - a_F)^{\frac{1}{\phi_F}} \left(\frac{C_{FR}}{c_R}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}} + \frac{1}{\phi_H C} f_R\right)
\end{aligned}$$

Using the above, we compute the inverse of the bilateral elasticities.

$$
\begin{aligned}
\Phi_{HF}^{-1} =& -\left(\frac{1}{c_H f_H}+\frac{1}{c_F f_F}\right)^{-1}\left[\left(\frac{f_{HH}}{f_H^2}-\frac{f_{HF}}{f_H f_F}\right)+\left(\frac{f_{FF}}{f_F^2}-\frac{f_{HF}}{f_H f_F}\right)\right] \\
=& -\left(\frac{1}{x_H f_H}+\frac{1}{x_F f_F}\right)^{-1} \\
& \times\left[\left(\frac{\frac{1}{\phi_H} f_H\left(-\frac{1}{c_H}+\frac{1}{C} f_H\right)}{f_H^2}-\frac{\frac{1}{\phi_H C} f_H f_F}{f_H f_F}\right)\right. \\
& \left.+\left(\frac{f_F\left(-\frac{1}{\phi_F c_F}+\frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}}+\frac{1}{\phi_H C} f_F\right)}{f_F^2}-\frac{\frac{1}{\phi_H C} f_H f_F}{f_H f_F}\right)\right] \\
=& -\left(\frac{1}{c_H f_H}+\frac{1}{c_F f_F}\right)^{-1} \\
& \times\left[\left(\frac{\left(-\frac{1}{\phi_H c_H}+\frac{1}{\phi_H C} f_H\right)}{f_H}-\frac{1}{\phi_H C}\right)\right. \\
& \left.+\left(\frac{\left(-\frac{1}{\phi_F c_F}+\frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}}+\frac{1}{\phi_H C} f_F\right)}{f_F}-\frac{1}{\phi_H C}\right)\right] \\
=& -\left(\frac{c_F f_F+c_H f_H}{c_H f_H c_F f_F}\right)^{-1} \\
& \times\left[-\frac{1}{\phi_H c_H f_H}+\frac{1}{\phi_H C}-\frac{1}{\phi_H C}-\frac{1}{\phi_F c_F f_F}+\frac{\phi_H-\phi_F}{\phi_H \phi_F f_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}}+\frac{1}{\phi_H C}-\frac{1}{\phi_H C}\right] \\
=& -\frac{c_H f_H c_F f_F}{c_F f_F+c_H f_H} \times\left[-\frac{1}{\phi_H c_H f_H}-\frac{1}{\phi_F c_F f_F}+\frac{\phi_H-\phi_F}{\phi_H \phi_F f_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{1}{C_{FR}}\right] \\
=& -\left[-\frac{c_F f_F}{c_F f_F+c_H f_H} \frac{1}{\phi_H}-\frac{c_H f_H}{c_F f_F+c_H f_H} \frac{1}{\phi_F}+\frac{c_H f_H}{c_F f_F+c_H f_H} \frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{c_F}{C_{FR}}\right] \\
=& \frac{c_F p_F}{P_{FR} C_{FR}}\left(\frac{c_F p_F+c_H p_H}{P_{FR} C_{FR}}\right)^{-1} \frac{1}{\phi_H}+\frac{c_H p_H}{PC} \frac{PC}{c_F p_F+c_H p_H} \frac{1}{\phi_F} \\
& -\frac{c_H f_H}{c_F f_F+c_H f_H} \frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{c_F}{C_{FR}} \\
=& \frac{c_F p_F}{PC} \frac{PC}{P_{FR} C_{FR}}\left(\frac{c_F p_F+c_H p_H}{PC} \frac{PC}{P_{FR} C_{FR}}\right)^{-1} \frac{1}{\phi_H} \\
& +\frac{c_H p_H}{PC} \frac{PC}{c_F p_F+c_H p_H} \frac{1}{\phi_F}-\frac{c_H f_H}{c_F f_F+c_H f_H} \frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{c_F}{C_{FR}} \\
=& \frac{\alpha_F}{1-\alpha_H}\left(\frac{\alpha_F+\alpha_H}{1-\alpha_H}\right)^{-1} \frac{1}{\phi_H}+\alpha_H\left(\alpha_F+\alpha_H\right)^{-1} \frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H} \frac{\phi_H-\phi_F}{\phi_H \phi_F} a_F^{\frac{1}{\phi_F}}\left(\frac{C_{FR}}{c_F}\right)^{\frac{1}{\phi_F}} \frac{c_F}{C_{FR}} \\
=& \frac{\alpha_F}{\alpha_F+\alpha_H} \frac{1}{\phi_H}+\frac{\alpha_H}{\alpha_F+\alpha_H} \frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H} \frac{\phi_H-\phi_F}{\phi_H \phi_F} \frac{p_F}{P_{FR}} \frac{c_F}{C_{FR}} \\
=& \frac{\alpha_F}{\alpha_F+\alpha_H} \frac{1}{\phi_H}+\frac{\alpha_H}{\alpha_F+\alpha_H} \frac{1}{\phi_F}-\frac{\alpha_H}{\alpha_F+\alpha_H} \frac{\phi_H-\phi_F}{\phi_H \phi_F} \frac{\alpha_F}{1-\alpha_H} \\
=& \frac{1}{\phi_H \phi_F\left(\alpha_F+\alpha_H\right)\left(1-\alpha_H\right)}\left[\alpha_F(1-\alpha_H) \phi_F+\alpha_H(1-\alpha_H) \phi_H-\alpha_H \alpha_F(\phi_H-\phi_F)\right] \\
\Phi_{HF}^{-1} =& \frac{\alpha_F \phi_F+\alpha_H \alpha_R \phi_H}{\phi_H \phi_F\left(\alpha_F+\alpha_H \alpha_R\right)}
\end{aligned}
$$

Hence:

$$\Phi_{HF}^{(H)} = \frac{\phi_H \phi_F \left(\alpha_F^{(H)} + \alpha_H^{(H)} \alpha_R^{(H)}\right)}{\alpha_F^{(H)} \phi_F + \alpha_H^{(H)} \alpha_R^{(H)} \phi_H}$$

We can compute $\Phi_{HR}^{(H)}$ in a similar fashion and obtain symmetrically:

$$\Phi_{HR}^{(H)} = \frac{\phi_H \phi_F (\alpha_R^{(H)} + \alpha_H^{(H)} \alpha_F^{(H)})}{\alpha_R^{(H)} \phi_F + \alpha_H^{(H)} \alpha_F^{(H)} \phi_H}$$

And it is easy to check that $\Phi_{FR}^{(H)} = \phi_F$.

E.2 Numerical Exercise with $N = 3$

To support the discussion in Section 4.1.3, we provide a concrete example here for a three-country setup, using the specification and elasticity formulas derived above in Appendix E.1. We impose steady-state asymmetries such that $Y_H = Y_F = 1$ and $Y_R = 0.5$. We consider in each country $n \in \{H, F, R\}$, a nested-CES specification where the aggregate consumption is a CES aggregate of the locally produced good, and a bundle of the two imported goods.

Suppose we want to set the parameters of these nested-CES aggregators so as to match the steady-state consumption shares and bilateral trade elasticities from a given parameterisation of the Kimball aggregator in order to replicate the first-order dynamics of the model. We can set the share parameters, $a_H^{(H)}$ and $a_F^{(H)}$, to match the steady-state consumption shares obtained from the Kimball aggregator directly. However, we now have two CES elasticity parameters, ϕ_H and ϕ_{FR}, to match the three bilateral elasticities.

Table E.1: Steady-State Expenditure Shares and Elasticities of Substitution: Nested CES vs. Kimball Aggregator

	Nested CES	Kimball ($\epsilon = 5$)
Expenditure Shares		
$\overline{\alpha}_H^{(H)}$	74.19	74.19
$\overline{\alpha}_F^{(H)}$	15.90	15.90
$\overline{\alpha}_R^{(H)}$	9.91	9.91
Matched Elasticities		
$\overline{\Phi}_{H,F}^{(H)}$	1.11	1.11
$\overline{\Phi}_{F,R}^{(H)}$	2.52	2.52
Derived Elasticity		
$\overline{\Phi}_{H,R}^{(H)}$	1.36	5.59
Nested CES parameters		
ϕ_H	0.88	NA
ϕ_{FR}	2.52	NA

The endowment asymmetry across our three countries implies asymmetric steady-state consumption shares and bilateral elasticities in country H, as stated in Table E.1. In other words, $\overline{\alpha}_F^{(H)} \neq \overline{\alpha}_R^{(H)}$. We are able to match country H's steady-state consumption shares, and two of its bilateral elasticities. However, we have no degree of freedom left to ensure that the third Kimball bilateral elasticity is matched by the nested-CES specification, and the nested-CES steady-state bilateral elasticity $\overline{\Phi}_{H,R}^{(H)}$ is not equal to the Kimball one. Consequently, a nested-CES specification is not flexible enough to match the first-order dynamics of our Kimball setup, due to the endowment asymmetry.